Kinematics

Coordinate Systems

- There are different ways to describe physical quantities, such as position, velocity, etc., depending on the nature of the coordinate system employed.
- Reference frames are basically the different perspectives of the viewer while the coordinate systems are the different ways to describe physical quantities in these perspectives.
- Two commonly used coordinate systems in biomechanics are the Cartesian coordinate system and the polar coordinate system:
- Cartesian Coordinates Polar Coordinates

Cartesian Coordinates

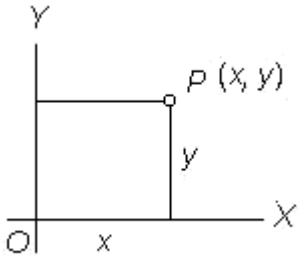


Figure 1

- In Cartesian coordinate system, one deals with a group of mutually perpendicular axes (or planes). Figure 1 shows the 2-D Cartesian coordinate system that involves 2 axes (or a plane). The *X* and *Y* axes shown in the figure are perpendicular to each other.
- Position *P* shown in Figure 1 can be described as (*x*, *y*). In other words, starting from the origin, one can reach *P* by moving in the direction of the *X* axis by *x* and then in the direction of the *Y* axis by *y*.
- *x* & *y* are the 2-D Cartesian coordinates. A rectangle was drawn for this purpose. For this reason, the Cartesian coordinate system is also called the rectangular coordinate system. Sequence of the movements is not important here.
- *x* & *y* shown in Figure 1 are called the components of the position and they are mutually independent. That is, a motion in the direction of the *X* axis is independent from a motion in the direction of the *Y* axis.

3-D Cartesian Coordinates

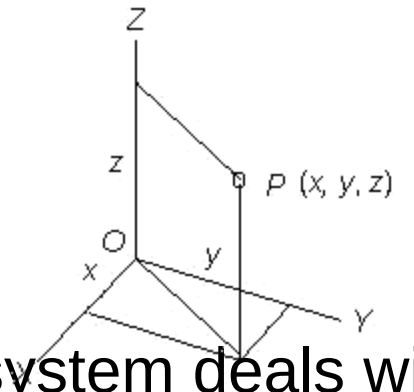


Figure 2

- The 3-D Cartesian coordinate system deals with 3 mutually perpendicular straight lines or axes (Figure 2). Again, one can describe the position of point *P* in terms of three Cartesian coordinates: *x*, *y* & *z*. Starting from the origin, one can reach point *P* by moving in the direction of the *X* axis by *x*, in the direction of the *Y* axis by *y*, and in the direction of the *Z* axis by *z*. The sequence here, again, does not matte
- Note in Figure 2 that the *Z* axis is aligned vertically. This is the traditional 3-D Cartesian coordinate system and will be used consistently throughout this website. Those who are accustomed to the ***ISB*** convention, in which the *Y* axis is aligned vertically, should switch the axes

Polar Coordinates

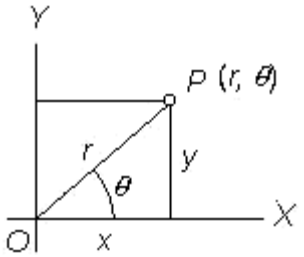


Figure 1

- The 2-D polar coordinate system involves the distance from the origin and an azimuth angle. Figure 1 shows the 2-D polar coordinate system, where *r* is the distance from the origin to point *P*, and is the azimuth angle measured from the horizontal (*X*) axis in the counterclockwise direction. Thus, the position of point *P* is described as (*r*,). *r* & are the 2-D polar coordinates.
- The polar coordinates are useful in describing the human body motion since the essence of the human body motion is the joint motions. The segments undergo rotations about the joint centers and using the azimuth angle (while *r* = const) in describing the body motion is more efficient than using the Cartesian coordinates. One thing to note here is that coordinates *r* & are not the same kind.

Theme: Kinematics of a point

- The task: For a table of plane coordinates (X and Y, mm) of a certain point on a human body during a physical exercise do kinematic analysis.
- A scale of a frame $\mu=0,02$ m/mm; $\Delta t=0,1$ s.

Frame number i	X	Y
	mm	
0	31	71
1	35	88
2	56	60
3	73	63
4	81	69
5	98	72
6	106	80
7	119	90
8	137	95
9	153	88
10	169	59

Possition (m)

- Calculate coordinates of a real position of the point (x and y, m).
- Plot the graph of x coordinate vs. time.
- Plot the graph of y coordinate vs. time.
- Plot the graph of the point trajectory.

Velocity (m/s)

- Calculate horizontal and vertical projections of a real velocity of the point.
- Plot the graph of v_x (horizontal projection of velocity) vs. time.
- Plot the graph of v_y (vertical projection of velocity) vs. time.
- Plot the graph of hodograph the point velocity.
- Plot the graph of magnitude of velocity magnitude vs. time.

vx	vy	v
m/s		
-0,9	7,9	8,0
2,5	-1,1	2,7
3,8	-2,5	4,5
2,5	0,9	2,7
2,5	0,9	2,7
2,5	1,1	2,7
2,1	1,8	2,8
3,1	1,5	3,4
3,4	-0,2	3,4
3,2	-3,6	4,8
3,2	-8,0	8,6

Acceleration (m/s^2)

- Calculate horizontal and vertical projections of acceleration of the point.
- Plot the graph of a_x (horizontal projection of acceleration) vs. time.
- Plot the graph of a_y (vertical projection of acceleration) vs. time.
- Plot the graph of hodograph the point acceleration.
- Plot the graph of magnitude of acceleration vs. time.

ax	ay	a
m/s2		
44,5	-128,0	135,5
23,5	-52,0	57,1
0,0	10,0	10,0
-6,5	17,0	18,2
0,0	1,0	1,0
-2,0	4,5	4,9
3,0	2,0	3,6
6,5	-10,0	11,9
0,5	-25,5	25,5
-1,0	-39,0	39,0
1,0	-49,0	49,0

$$r = \sqrt{x^2 + y^2}$$

$$\theta = \begin{cases} \tan^{-1}\left(\frac{y}{x}\right) & (x > 0) \\ \tan^{-1}\left(\frac{y}{x}\right) + \pi & (x < 0, y > 0) \\ \tan^{-1}\left(\frac{y}{x}\right) - \pi & (x < 0, y < 0) \\ \frac{\pi}{2} & (x = 0, y > 0) \\ -\frac{\pi}{2} & (x = 0, y < 0) \end{cases}$$

The relationship between the 2-D Cartesian coordinates and the 2-D polar coordinates can be summarized as:

$$x = r \cdot \cos\theta$$

$$y = r \cdot \sin\theta$$

$$-\pi \leq \theta < \pi$$

coordinates in the sense of the polar coordinates, as shown on Figure 2.

X_r X_θ

axis is the tangential axis. Using the *P*- system (using the Cartesian system in the ser

X_r X_θ

3-D (Spherical) Polar Coordinates

pherical coordinate system involves the distance from the origin and 2 angles (Figure 3).

ϕ θ
,), where *r* = the distance from the origin (*O*), ϕ = the horizontal azimuth angle measur

ationship between the spherical coordinates and the Cartesian coordinates can be summar

$\sqrt{x^2+z^2}$

$\left(\frac{y}{x}\right) \quad (x>0)$

$\left(\frac{y}{x}\right)+\pi \quad (x<0, y>0)$

$\left(\frac{y}{x}\right)-\pi \quad (x<0, y<0)$

$(x=0, y>0)$

$(x=0, y<0)$

$\left(\frac{\sqrt{x^2+y^2}}{z}\right) \quad (z>0)$

$\left(\frac{\sqrt{x^2+y^2}}{z}\right)+\pi \quad (z<0)$ **[3]**

$(z=0)$

$-\pi \le \phi < \pi$ **and** $0 \le \theta \le \pi$ **. Or**

$\theta) \cdot \cos\phi = r \cdot \cos\phi \cdot \sin\theta$

$\theta) \cdot \sin\phi = r \cdot \sin\phi \cdot \sin\theta$ **[4]**

te system shown in Figure 2, one can also define a moving 3-D Cartesian system. See Figu

shown in Figure 4 is useful in describing the motion of point P in terms of the radial motion
the unit vector) of the axes can be described in those of the global reference frame. Let the

$\mathbf{e}_r$ $\mathbf{e}_\theta$ $\mathbf{e}_\phi$

Vector & Matrix

- Physical quantities that have both magnitude and associated direction are called **vectors**. Examples of vector include displacement, velocity, acceleration, force, momentum, and impulse. Quantities such as temperature, mass, time, energy, power, and work, on the other hand, do not require directional info to completely describe them and these quantities are called **scalars**. Although the vector algebra is considered to be somewhat more complicated than that of the scalars, vectors are very convenient tools in describing and explaining the motions of bodies.
- Another mathematical tool commonly used in mechanics is the **matrix**. Matrices are mainly used in axis transformation and 3-dimensional angular kinematics and kinetics in motion analysis.

Vector

ce type to distinguish them from scalars. Graphically speaking, an arrow can conveniently a

ents:

termine the dimension of a vector: 2-dimensional (2 components) vs. 3-dimensional (3 com

Figure 2

phically (using arrows) and with components.
the length of the vector arrow shown in Figure 2:

alled a **unit vector**. The unit vector of vector **v** shown in Figure 2 can be obtained by dividin

vectors) can be expressed as

Z axes, respectively.

[9]

e. One can move the vectors freely to connect tip-to-tale as far as the magnitudes and the
gnifies (Figure 4):

10]

thens when $|d| > 1$ while it shortens if $|d| < 1$. If d is negative, the direction of the vector reve

[11]

[12]

- In analyzing an object's motion, we need to focus our attention to certain portion (a part or the whole) of the object which we conceptually separate it from the environment external to it. This specific portion subject to analysis is called as **the mechanical system**. For example, if one is interested in analyzing motion of the right arm of a subject, the subject's right arm is the mechanical system. The rest of the body (the head, left arm, trunk and the lower extremities) are all **external** to the mechanical system. The right hand, right forearm and the right upperarm that compose the right arm are **internal** to the system. Identifying the mechanical system is important in the sense that it will allow the analyst to distinguish **the external forces**, the forces acting on the system from outside of the system, from **the internal forces**, forces acting within the system among the elements composing the system. The linear motion of the center of mass of the system is affected only by the external forces. Therefore, the first task in motion analysis is to define the mechanical system.

Motion of a Particle

cal systems: a **particle**, a **system of particles**, a **rigid body**, and a **system of [linked] rig**

tion of motion'. while **kinetics** is 'the study of explanation of motion'. **Description** focuses o

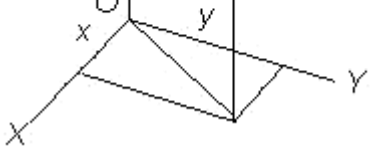


Figure 1

$$\mathbf{r} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z \quad \textbf{[1]}$$
$$= (x, y, z)$$

, and x, y & z = the Cartesian coordinates. One may also use a polar coordinate system (Fi

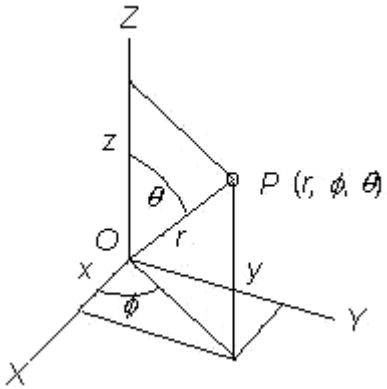


Figure 2

$$\mathbf{r} = (r, \phi, \theta) \quad \textbf{[2]}$$

From [1] & [2]:

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\phi = \begin{cases} \tan^{-1}\left(\dfrac{y}{x}\right) & (x > 0) \\ \tan^{-1}\left(\dfrac{y}{x}\right) + \pi & (x < 0, y > 0) \\ \tan^{-1}\left(\dfrac{y}{x}\right) - \pi & (x < 0, y < 0) \\ \dfrac{\pi}{2} & (x = 0, y > 0) \\ -\dfrac{\pi}{2} & (x = 0, y < 0) \end{cases}$$

$$\begin{cases} \tan^{-1}\left(\dfrac{\sqrt{x^2 + y^2}}{z}\right) & (z > 0) \end{cases}$$

position or the rate of displacement. From Figure 3:

Figure 3

y taking time-derivatives of the coordinates as shown in [6]. The direction of the instantaneo

in velocity or:

[7]

first time-derivative of position, and acceleration is the second time-derivative of position. T

cles inter-related. The equations for a system of particles can be readily used to develop tho

ch as CM and momentum are generally very simple. But those for the angular motion variab

Center of Mass

article is defined as the product of its mass and position vector (Figure 1):

Figure 1

ent, *m* = the mass, and **r** = the position vector. The total mass moment of the system of part

Figure 2

hich suffices the following relationship:

$_{CM}$ = the position vector of the CM. In other words, the CM is a single point which effectively

[4]

as

Figure 3

[5]

ass moment of the system with respect to its CM. From [3] and [5]:

moments of the particles with respect to their CM always becomes zero.

Rigid Body Dynamics

- For analytical convenience, human body segments are considered as rigid bodies. A rigid body is similar to a system of particles in the sense that it is composed of particles. Therefore most of the equations for a system of particles are usable in the dynamics of a rigid body. The main difference is of course that the rigid body is **rigid**. There is no migration of mass within a rigid body. As a result, the relative positions among the particles composing a rigid body do not change. This will further simplify the equations obtained from a system of particles.
- The main strategy in analyzing the motion of a rigid body is to split the motion into the linear motion of the CM and the angular motion of the body about its CM. This is because all particles composing the body show the same relative angular motion about the CM. In other words, one can describe the motion of the rigid body as a whole rather than those of the particles individually. The physical characteristics of a rigid body can be described by its inertial properties: mass and moment of inertia. For this reason, a major portion of discussion will be dedicated to the inertial properties of the rigid body: moment of inertia & inertia tensor.

Moment of Inertia of a Systems of Particles

f the system and its angular counterpart is the so-called **moment of inertia**. The moment o
f particles in which the relative positions of the particles do not change. The moment of iner

[1]

where m = the mass of the particle, and r = the shortest distance from the axis of rotation to the particle (Figure 1).

$$I = mr^2$$

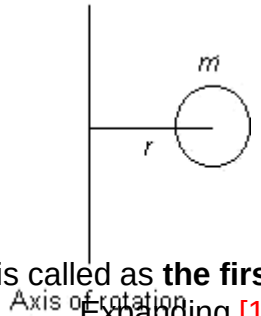


Figure 1

mass moment. Mass times distance, *m***r**, is called as **the first mass moment**. This concept of first mass moment is normally used in de

Expanding [1] for a system of particles:

[2]

$$I = \sum_i m_i r_i^2$$

Moment of Inertia of a Rigid Body

one can obtain the moment of inertia of a rigid by shown in Figure 2:

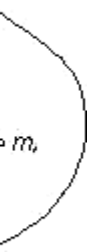


Figure 2

[3]

n of particle *i*, and **n** = the unit vector of the axis of rotation. Note here that the axis of rotation passes through the local reference frame, t

Calculation of the MOI

ent of inertia, one can replace the summation shown in [2] of Inertia Tensor by an integratio

r distance from the particle to the axis of rotation, and *dm* = the mass of the particle which i

n uniform rod shown in Figure 1. The MOI of the rod about the *Y* axis is

Figure 1

[2]

e mass of the rod, and *L* = the length of the rod.

Circular Ring

ment of inertia of the uniform circular ring shown in Figure 2 about the *Z* axis (the symmetry

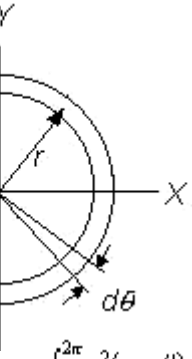


Figure 2

$$m = \int_0^{2\pi} r^2(\rho \cdot dl)$$

[4]

$$\cdot d\theta) = \rho r^3 \int_0^{2\pi} d\theta = 2\pi\rho r^3 = mr^2$$

[5]

he length of the arc formed by $d\theta$. [5] is also applicable to a circular cylinder.

e considered as a cascade of uniform circular rings as shown in Figure 3. Thus, from [5], the

Figure 3

5]

its longitudinal axis as well.

d as a cascade of uniform circular discs as shown in Figure 4. From [6], the moment of iner

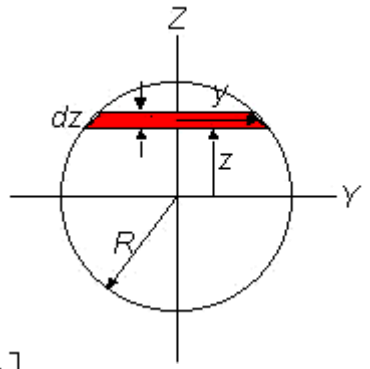


Figure 4

$$I_{zz} = \int_{-R}^{R}\left[\frac{1}{2}\pi\rho y^4\right]\cdot dz$$

$$= \int_{-R}^{R}\left[\frac{1}{2}\pi\rho\left(R^2 - z^2\right)^2\right]\cdot dz = \frac{1}{2}\pi\rho\int_{-R}^{R}\left[R^4 - 2R^2z^2 + z^4\right]\cdot dz$$

$$= \frac{1}{2}\pi\rho\left[R^4 z - \frac{2}{3}R^2z^3 + \frac{1}{5}z^5\right]_{-R}^{R} = \frac{8}{15}\pi\rho R^5 = \frac{2}{5}mR^2 \qquad [8]$$

since

$$m = \frac{4}{3}\pi\rho R^3 \qquad [9]$$

Perpendicular-Axis Theorem

sc (Figure 3). The moment of inertia of a uniform circular disc about its perpendicular axis (

equal to sum of those about the *X* and *Y* axes. [10] is true for any rigid lamina: the MOI of a
e,

Equations for this approach.

ular column. Circular bar can be regarded as a cascade of circular discs as shown in Figur

Parallel-Axis Theorem

Figure 5

[15]

he disc about the *Y'* axis. [16] basically says that the MOI of a circular plate about the *Y* axis is equal to the sum of the MOI about the pa

dz

[17]

OI between a thin rod and a thick rod (circular column). Similarly, [13] and [17] shows the difference in the MOI between a circular disc a

ssible only when the body has a known geometric shape. In the mathematical human body
ion approach has not much use and a direct measurement must be attempted. Figure 6 sh

Physical Pendulum & Direct Measurement

Figure 6

dy about the *X* axis is then

MOI of the body about the *X* axis, α = angular acceleration, *m* = the mass of the body, *g* = t

e phase angle, *T* = the period of the pendulum. As shown in [22], the MOI of the body about the *X* axis, after all, can be computed from th

- **References and Related Literature**
- **Chandler, R. F., Clauser, C. E., McConville, J. T., Reynolds, H. M. and Young, J. W. (1975)**. *Investigation of inertial properties of the human body*. AMRL-TR-74-137, AD-A016-485. DOT-HS-801-430. Aerospace Medical Research Laboratories, Wright-Patterson Air Force Base, Ohio.
- **Hanavan, E. P. (1964)**. *A mathematical model of the human body*. AMRL-TR-64-102, AD-608-463. Aerospace Medical Research Laboratories, Wright-Patterson Air Force Base, Ohio.
- **Yeadon, M. R. (1990)**. The simulation of aerial movement-II. A mathematical inertia model of the human body. *J. Biomechanics* **23**, 67-74.

- In was mentioned earlier that the human body segments are considered as rigid bodies for analytical convenience. Since the human body is composed of several segments (14 - 15, depending on the model used), it can be considered as a **system of rigid bodies**. In spite of some similarities, a system of rigid bodies is quite different from a system of particles in the sense that the elements of the system are rigid bodies that have not only the mass but also the volume (moment of inertia). Therefore we need to develop a new set of equations dedicated to a system of rigid bodies.
- Some of the mechanical quantities of a system of rigid bodies can be directly computed by summing those of the segments, such as the momentum and the kinetic energy. But some other quantities deserve more attention. The pages included in this section are
- Whole Body Center of Mass Whole Body MOI Whole Body Angular Momentum
- The human body is in fact more than a simple collection of rigid bodies. The segments are inter-linked at the joints and, for this reason, the human body is often called as **the linked-segment system**. The fact that the segments interact each other adds additional complexity to the system and the related issues are covered in a separate section in Volume II: Theories and Practices in Motion Analysis.

Figure 1

egment CM, **R** = the position of the whole body CM, and $\mathbf{r}_i$ = the relative position of the seg

[2]

e body center of mass is the balance point of the system and must suffice

[4]

ment CM.

nt CM to the whole body CM. Therefore:

- Human body is a system of rigid segments inter-linked at the joints. One of the important aspects of this type of system is the interaction between the inter-linked segments at the joints. The joints (ligaments, joint capsule, etc.) themselves are passive structures that only apply constraining forces, when they are stretched, on to the bony structures involved in the joints. The active components of the system are the skeletal muscles that produces forces through voluntary contraction.
- It is possible to assess the interaction between the segments through the joint & muscle and the work done by the muscles through a procedure called **inverse dynamics**. Inverse dynamics literally means that one tries to figure out the cause of motion based on the effects (motion) and the inertial properties of the object in motion. One can quantify the human (or animal) motion through motion analysis. The inertial properties of the segments can be obtained from direct or indirect methods.
- Human body is best described as **the linked-segment system**. The basic assumptions here are:
- The joints are all **pin joints** and there is no friction at the joint. Thus, the forces produced by the elements of the joint such as ligaments and joint capsules are all **concentric** about the joint centers. In other words, all these forces pass through the joint centers. Muscles are the only elements that can produce **eccentric forces** about the joint centers. All muscles are **uni-articular** and there is no intervening structure that can act as pulley. In other words, muscles are straight and directly attach to the segments.
- The following pages in this section explain the inverse dynamics procedure, and the joint energetics:
- Joint Torque Joint Torque Revisited Data Interpretation Joint Energetics
- Although the inverse dynamics approach is fairly useful, it has its own limits. One must clearly understand these limits to prevent misinterpretation of the calculation results.

Direct Linear Transformation (DLT)