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On the Regular Behavior of Entire Functions of Zero Order along Curves of Regular Rotation

N. V. Zabolotskii[1]*, M. R. Mostova[2], and V. E. Shvets[1]*****

[1] *Ivan Franko National University of Lviv, Lviv, 79000 Ukraine*

[2] *Lviv State University of Physical Culture named after Ivan Bobersky, Lviv, 79000 Ukraine*

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Abstract—We study the relationship between the existence of an v-density $\Delta^\gamma(\alpha,\beta)$ of roots along the curves of regular rotation of an entire function of zero order and the regular behavior at infinity of its logarithm.

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1. INTRODUCTION

Questions related to the study of the behavior of entire functions over infinite curves (in particular, logarithmic spirals) were considered in [1]–[5]. Macintyre [1] introduced the concept of indicator along a logarithmic spiral and generalized the concept of associated function; Kennedy [2] studied analytical functions in domains bounded by spiral-like curves C_k, $k=1,2$, $\arg z=\psi_k(z)$, $z\in C_k$, where $\psi_k(r)$ is a continuous, almost everywhere differentiable function for $r>0$ and $r|\psi'(r)|\le \text{const}$; Balashov [3] and Heifits [4] proved, respectively, theorems of Valiron and Valiron–Titchmarsh type for positive-order entire functions with zeros on a logarithmic spiral.

Denote

$$l^\gamma_\varphi(a,b)=\{z=|z|\exp(i(\varphi+\gamma(|z|))):a\le|z|\le b\},\qquad l^\gamma_\varphi(1,+\infty)=l^\gamma_\varphi,$$

where $\varphi\in\mathbb{R}$ and $\gamma(r)$ is a real-valued differentiable function on $[a,b]$. Following Balashov [3], we call l^γ_φ a *curve of regular rotation* if the limit

$$\lim_{r\to+\infty} r\gamma'(r)=c,\qquad -\infty<c<+\infty$$

exists. In the case $\gamma(r)=c\ln r$, we obtain the logarithmic spiral l^c_φ. Note that a curve of regular rotation differs significantly from a ray if there exists no finite limit $\lim_{r\to+\infty}\gamma(r)$.

In [5], the main theorem of the theory of entire functions of completely regular growth of finite positive order was generalized; this theory was generalized to curves of regular rotation; this theory was constructed by Levin and Pfluger (see [6, Chaps. 2, 3]); and, in [7], an analogue of this theorem was proved for entire functions of zero order, which form an important and (in certain applications) popular subclass of entire functions. It is easy to show that

$$f(z)=\prod_{n=1}^{+\infty}\left(1-\frac{z}{a^n}\right),\qquad a=|a|e^{i\alpha},\quad |a|>1,\quad \alpha\in[0,2\pi),$$

*E-mail: mykola.zabolotskyy@lnu.edu.ua
**E-mail: mariana.mostova@gmail.com
***E-mail: shwe@ukr.net

is an entire function of zero order whose roots have an angular v-density with respect to the comparison function $v(r)=\ln r$ and, in the case of the incommensurability of the numbers α and π, they are uniformly distributed over the angles ($d\Delta(\psi)=(\Delta_1/(2\pi))\,d\psi$, $\Delta_1=1/\ln|a|$). By Theorem 1 from [7] the following equality

$$\ln f(z)=\frac{1}{2}\,\Delta_1\ln^2 r+o(\ln r),\qquad r\to+\infty,$$

holds outside some C_0-set of values $z=re^{i\varphi}$, $0\le\varphi<2\pi$. In fact, in this case, the roots of f lie on the logarithmic spiral l_0^c ($c=\alpha/\Delta_1$), and it is natural to study the behavior of $\ln f$ along the logarithmic spirals l_φ^c. Using results from [7] (see Theorem 1), outside some C_0-set of values $z=re^{i(\varphi+c\ln r)}$, $0<\varphi<2\pi$, we obtain the asymptotic representation

$$\ln f(z)=\frac{1}{2}\,\Delta_1(1+ic)\ln^2 r+i\Delta_1(\theta-\pi)\ln r+o(\ln r),\qquad r\to+\infty,$$

which is more informative than the behavior of $\ln f$ over rays.

The example of this special function f corroborates the importance of conducting studies of the relationship between the distribution of roots along curves of regular rotation of entire functions of zero order and the regular behavior at infinity of their logarithms along such curves. The theorems obtained in the present paper generalize results contained in [7].

2. DEFINITIONS AND STATEMENT OF THE MAIN RESULTS

Let L be the class of continuously differentiable, positive, nondecreasing, unbounded functions v on $[0,+\infty)$ such that $rv'(r)/v(r)\to 0$ as $r\to+\infty$. It is well known that, up to equivalent functions, the class L coincides with the class of slowly increasing functions $\beta(r)$ such that $\beta(2r)\sim\beta(r)$, $r\to+\infty$. Also note that the functions $v\in L$ are of zero order, namely, $\lim_{r\to+\infty}(\ln^+ v(r)/\ln r)=0$.

Let $H_0(v)$ denote the class of entire functions of zero order such that the counting function $n(r)=n(r,0,f)$ of their roots satisfies the condition $n(r)=O(v(r))$, $r\to+\infty$. For $0<\beta-\alpha\le 2\pi$, we put

$$D^\gamma(r;\alpha,\beta)=\bigcup_{\alpha<\varphi\le\beta} l_\varphi^\gamma(1,r),$$

and $n^\gamma(r;\alpha,\beta)$ is the number of roots $f\in H_0(v)$ in the curvilinear sector $D^\gamma(r;\alpha,\beta)$.

We will say that the set of roots of a function $f\in H_0(v)$ *has* an v-*density* $\Delta^\gamma(\alpha,\beta)$ *along the curves of regular rotation* l_φ^γ if, for all $\alpha,\beta\in\mathbb{R}$, with the possible exception of a countable set of values of α and β, the following limit exists:

$$\lim_{r\to+\infty}\frac{n^\gamma(r;\alpha,\beta)}{v(r)}=\Delta^\gamma(\alpha,\beta).$$

In this case, for a fixed φ_1, the equality $\Delta^\gamma(\varphi)-\Delta^\gamma(\varphi_1)=\Delta^\gamma(\varphi_1,\varphi)$ defines on $(\varphi_1,\varphi_1+2\pi]$, up to a constant, a nondecreasing function $\Delta^\gamma(\varphi)$, which we will extend to the whole set $\mathbb{R}$ by the rule

$$\Delta^\gamma(\varphi+2\pi)-\Delta^\gamma(\varphi)=\Delta^\gamma(\varphi_1+2\pi)-\Delta^\gamma(\varphi_1),\qquad \varphi\in\mathbb{R}.$$

A curve of regular rotation l_θ^γ will be called *ordinary* for $f\in H_0(v)$ if

$$\lim_{\varepsilon\to 0+}\varlimsup_{r\to+\infty}\frac{n^\gamma(r;\theta-\varepsilon,\theta+\varepsilon)}{v(r)}=0.$$

All the other curves of regular rotation l_φ^γ will be called *exceptional*. If the roots of $f\in H_0(v)$ have an v-density $\Delta^\gamma(\alpha,\beta)$ along the curves of regular rotation l_φ^γ, then, from the monotonicity of the function $\Delta^\gamma(\varphi)$, it follows that the set of exceptional curves of regular rotation l_φ^γ for f is at most countable.

Without loss of generality, we assume that $f(0)=1$. By $\ln(1-z/a_n)$, $a_n\in l_\theta^\gamma$, we will denote the univalent branch in the domain $D(l_\theta^\gamma)=\mathbb{C}\setminus l_\theta^\gamma(|a_n|,+\infty)$ of the multivalued function $\mathrm{Ln}(1-z/a_n)$ such that $\ln(1-z/a_n)|_{z=0}=0$. Then, for

$$f(z)=\prod_{n=1}^{+\infty}\biggl(1-\frac{z}{a_n}\biggr)\in H_0(\upsilon),\qquad \upsilon\in L, \tag{2.1}$$

we obtain

$$\ln f(z)=\sum_{n=1}^{+\infty}\ln\biggl(1-\frac{z}{a_n}\biggr),\qquad z\in\mathbb{C}\setminus\biggl(\bigcup_{n=1}^{+\infty}l_{\varphi_n}^\gamma(r_n,+\infty)\biggr),$$

where $r_n=|a_n|$ is the smallest modulus of the root f that lies on the curve of regular rotation $l_{\varphi_n}^\gamma$, $\varphi_n=\arg a_n\in[-\pi,\pi)$. For $-\pi\le\psi<\pi$, we denote by $\widehat{h}(\theta;\psi)$ the 2π-periodic extension of the function $h(\theta;\psi)=\theta-\psi-\pi$, $\theta\in(\psi,\psi+2\pi)$, to $\mathbb{R}$. For ordinary curves of regular rotation l_θ^γ of a function $f\in H_0(\upsilon)$, $\upsilon\in L$, we let

$$H_f^\gamma(\theta)=\int_{\theta-2\pi}^{\theta}(\theta-\psi-\pi)\,d\Delta^\gamma(\psi)=\int_{-\pi}^{\pi}\widehat{h}(\theta;\psi)\,d\Delta^\gamma(\psi),$$

$$N^\gamma(r)=N^\gamma(r,0,f)=\int_{l_\theta^\gamma(1,r)}\frac{n(|w|)}{w}\,dw=\int_1^r\frac{n(t)(1+it\gamma'(t))}{t}\,dt.$$

Following Levin [6, p. 119], we will call the quantity

$$\rho^*(E)=\varlimsup_{r\to+\infty}\biggl(\frac{1}{r}\sum_{|c_j|\le r}r_j\biggr)$$

the *upper linear density of the set E of disks* $\{z\colon|z-c_j|<r_j\}$, $j\in\mathbb{N}$, and the set of zero upper linear density is called a C_0-*set*.

Theorem 1. *Let $\upsilon\in L$, $f\in H_0(\upsilon)$, and let the roots of f have υ-density $\Delta^\gamma(\alpha,\beta)$ along the curves of regular rotation l_φ^γ. Then there exists a C_0-set E such that, for all the ordinary curves of regular rotation l_θ^γ of the function f,*

$$\ln f(z)=N^\gamma(r)+iH_f^\gamma(\theta)\upsilon(r)+o(\upsilon(r)),\qquad z=re^{i(\theta+\gamma(r))}\notin E,\quad r\to+\infty. \tag{2.2}$$

Let $\Gamma_m=\bigcup_{j=1}^m l_{\theta_j}^\gamma$, $-\pi\le\theta_1<\theta_2<\dots<\theta_m<\theta_{m+1}=\pi$, be a finite system of curves of regular rotation.

Remark 1. If the roots of $f\in H_0(\upsilon)$ lie on Γ_m, then the existence of a υ-density $\Delta^\gamma(\alpha,\beta)$ along the curves of regular rotation l_φ^γ is equivalent to the relations ($\Delta_j\ge0$, $j=1,\dots,m$)

$$n(r;\theta_j)=\Delta_j\upsilon(r)+o(\upsilon(r)),\qquad r\to+\infty,$$

where $n(r;\theta_j)$ is the counting function of the roots of f on $l_{\theta_j}^\gamma$. In this case,

$$H_f^\gamma(\theta)=\sum_{j=1}^m\Delta_j\widehat{h}(\theta;\theta_j),\qquad\theta\ne\theta_j,\quad j=1,\dots,m.$$

Theorem 2. *Let $\upsilon\in L$, let $f\in H_0(\upsilon)$, let the roots of f lie on Γ_m, and let*

$$G(\theta)=G(\theta;\Gamma_m)=\sum_{j=1}^m\Delta_j\widehat{h}(\theta;\theta_j),\qquad\Delta_j\ge0,\quad\theta\in[-\pi,\pi).$$

If, for an arbitrary $\delta > 0$,

$$\theta \in [-\pi, \pi] \setminus \bigcup_{j=1}^{m+1} (\theta_j - \delta, \theta_j + \delta),$$

the following relation holds uniformly in:

$$\ln f(re^{i(\theta+\gamma(r))}) = N^\gamma(r) + iG(\theta)\upsilon(r) + o(\upsilon(r)), \qquad r \to +\infty, \tag{2.3}$$

then the roots of f *have an* υ*-density* $\Delta^\gamma(\alpha, \beta)$ *along the curves of regular rotation* l_θ^γ.

Remark 2. It was shown in [8] that, in the general case of an arbitrary arrangement of the roots of $f \in H_0(\upsilon)$, $\upsilon \in L$, the converse statement to Theorem 1 does not hold even in the case when the curves of regular rotation are rays.

3. AUXILIARY RESULTS

In this section, we present six Lemmas that will be used in the proof of the theorems.

Lemma 1. *Suppose that* $\upsilon \in L$, $l_{-\pi}^\gamma$ *is a curve of regular rotation,* $0 < \delta < 1$, $\beta(t) - \gamma(t) \to 0$ *as* $t \to +\infty$, $\alpha(t)$ *is a piecewise continuous function on* $[1, +\infty)$, $\alpha(t) \to 0$ *as* $t \to +\infty$. *Then, for* $z = re^{i(\varphi+\beta(r))}$, $-\pi < \varphi < \pi$,

$$J_1 = \int_{l_\varphi^\gamma(1,r)} \frac{\alpha(|w|)\upsilon(|w|)}{w - z}\, dw = o(\upsilon(r)), \qquad r \to +\infty, \tag{3.1}$$

$$J_2 = z \int_{l_\varphi^\gamma(r,+\infty)} \frac{\alpha(|w|)\upsilon(|w|)}{w(w - z)}\, dw = o(\upsilon(r)), \qquad r \to +\infty, \tag{3.2}$$

and relations (3.1), (3.2) *hold uniformly with respect to* $\varphi \in [-\pi + \delta, \pi - \delta]$.

Proof. Let $\varepsilon > 0$ be an arbitrary number, and let $K_1, K_2, \dots$ be positive constants. We put

$$\eta = e^{-\delta/(4|c|)}, \qquad \text{where} \quad c = \lim_{t \to +\infty} (t\gamma'(t)).$$

If $c = 0$, then we assume that $\eta = 1/2$.

For an arbitrary $b > 0$, we have [3],

$$\lim_{x \to +\infty} (\gamma(bx) - \gamma(x)) = c \ln b,$$

it follows that, for $t \in [\eta r, r/\eta]$,

$$\widetilde{\varphi} = \varphi + \beta(r) - \gamma(t) = \varphi + (\beta(r) - \gamma(r)) + (\gamma(r) - \gamma(t)) \to \varphi + c \ln b^*$$

as $r \to +\infty$, $b^* \in [\eta, 1/\eta]$. For $\varphi \in [-\pi + \delta, \pi - \delta]$, for $r \geq r_1$, we have

$$\widetilde{\varphi} \geq (\varphi + c \ln b^*) - \frac{\delta}{2} \geq -\pi + \frac{\delta}{2} - |c \ln \eta| = -\pi + \frac{\delta}{2} - |c| \ln e^{\delta/(4|c|)} = -\pi + \frac{\delta}{4},$$

$$\widetilde{\varphi} \leq (\varphi + c \ln b^*) + \frac{\delta}{2} \leq \pi - \frac{\delta}{2} + |c| \ln \frac{1}{\eta} = \pi - \frac{\delta}{4},$$

and hence $|\widetilde{\varphi}| \leq \pi - \delta/4$. In view of the inequality $|t + re^{i\theta}| \geq (t + r)\sin(\delta_1/2)$ (see, e.g., [9]), for $|\theta| \leq \pi - \delta_1$, $0 < \delta_1 < 1$, when $t \in [\eta r, r/\eta]$, we obtain

$$|w - z| = |te^{i(-\pi+\gamma(t))} - re^{i(\varphi+\beta(r))}| = |t + re^{i\widetilde{\varphi}}| \geq (t + r)\sin\left(\frac{\delta}{8}\right).$$

We have $w = te^{i(-\pi+\gamma(t))}$, $dw = (1 + it\gamma'(t))e^{i(-\pi+\gamma(t))}\, dt$,

$$|J_1| = \left| \left(\int_1^{\eta r} + \int_{\eta r}^{r} \right) \frac{\alpha(t)\upsilon(t)(1 + it\gamma'(t))}{w - z}\, e^{i(-\pi+\gamma(t))}\, dt \right|$$

$$\le K_1\int_1^{\eta r}\frac{|\alpha(t)|\upsilon(t)}{|z|-|w|}\,dt+K_2\int_{\eta r}^{r}\frac{|\alpha(t)|\upsilon(t)\,dt}{(t+r)\sin(\delta/8)}$$
$$\le\frac{K_1\upsilon(r)}{r(1-\eta)}\int_1^r|\alpha(t)|\,dt+\frac{K_2\upsilon(r)}{r(1+\eta)\sin(\delta/8)}\int_1^r|\alpha(t)|\,dt<\varepsilon\upsilon(r),\qquad r\ge r_2,$$

because $\int_1^r|\alpha(t)|\,dt=o(r)$, $r\to+\infty$.

We put $\alpha^*(r)=\sup\{|\alpha(t)|\colon r\le t\le r/\eta\}$. Since

$$\upsilon(r/\eta)\sim\upsilon(r),\quad \alpha^*(r)\to 0,\quad \int_r^{+\infty}|\alpha(t)|\upsilon(t)/t^2\,dt=o(\upsilon(r)/r)\qquad\text{as}\quad r\to+\infty,$$

it follows that

$$|J_2|\le K_3r\int_r^{r/\eta}\frac{|\alpha(t)|\upsilon(t)\,dt}{t(t+r)\sin(\delta/8)}+K_4r\int_{r/\eta}^{+\infty}\frac{|\alpha(t)|\upsilon(t)}{t(t-r)}\,dt$$
$$\le K_5\alpha^*(r)\frac{\upsilon(r)}{r}\int_r^{r/\eta}dt+K_6r\int_r^{+\infty}\frac{|\alpha(t)|\upsilon(t)}{t^2}\,dt$$
$$<K_7\alpha^*(r)\upsilon(r)+\frac{\varepsilon}{2}\upsilon(r)<\varepsilon\upsilon(r),\qquad r\ge r_3.$$

Lemma 1 is proved. □

Let $k\in\mathbb{N}\cup\{0\}$, $\widetilde{\upsilon}(t)=t\upsilon'(t)$, $\upsilon(1)=0$, $t\gamma'(t)=c+\varepsilon(t)$, $\varepsilon(t)\to 0$ as $t\to+\infty$. We put

$$A_k(\tau,\upsilon)=\int_1^{\tau}\upsilon(t)t^ke^{i(k+1)\gamma(t)}\,dt,\qquad B_k(\tau,\upsilon)=\int_{\tau}^{+\infty}\upsilon(t)t^{-k-2}e^{-i(k+1)\gamma(t)}\,dt.$$

Lemma 2. *If $\upsilon\in L$, then*

$$A_k(\tau,\upsilon)=\frac{1}{1+ic}\left(\frac{\upsilon(\tau)\tau^{k+1}e^{i(k+1)\gamma(\tau)}}{k+1}-\frac{1}{k+1}A_k(\tau,\widetilde{\upsilon})-iA_k(\tau,\varepsilon\cdot\upsilon)\right),\tag{3.3}$$
$$B_k(\tau,\upsilon)=\frac{1}{1+ic}\left(\frac{\upsilon(\tau)\tau^{-k-1}e^{-i(k+1)\gamma(\tau)}}{k+1}+\frac{1}{k+1}B_k(\tau,\widetilde{\upsilon})-iB_k(\tau,\varepsilon\cdot\upsilon)\right).\tag{3.4}$$

Proof. Integrating by parts, we obtain

$$A_k(\tau,\upsilon)=\frac{\upsilon(t)t^{k+1}}{k+1}e^{i(k+1)\gamma(t)}\Big|_1^{\tau}-\frac{1}{k+1}\int_1^{\tau}t^{k+1}e^{i(k+1)\gamma(t)}(\upsilon'(t)+i(k+1)\gamma'(t)\upsilon(t))\,dt$$
$$=\frac{\upsilon(\tau)\tau^{k+1}e^{i(k+1)\gamma(\tau)}}{k+1}$$
$$-\frac{1}{k+1}\int_1^{\tau}(t\upsilon'(t))t^ke^{i(k+1)\gamma(t)}\,dt-i\int_1^{\tau}\upsilon(t)(t\gamma'(t))t^ke^{i(k+1)\gamma(t)}\,dt$$
$$=\frac{\upsilon(\tau)\tau^{k+1}e^{i(k+1)\gamma(\tau)}}{k+1}-\frac{1}{k+1}A_k(\tau,\widetilde{\upsilon})-icA_k(\tau,\upsilon)-iA_k(\tau,\varepsilon\cdot\upsilon),$$

from which follows (3.3). Relation (3.4) is obtained similarly. Lemma 2 is proved. □

Lemma 3. *Let $\upsilon\in L$, $w\in l^{\gamma}_{-\pi}$, $\beta(r)-\gamma(t)\to 0$ as $r\to+\infty$, $0<\delta<1$. Then, for $z=re^{i(\varphi+\beta(r))}$, as $r\to+\infty$, the following relations hold uniformly with respect to $\varphi\in[-\pi+\delta,\pi-\delta]$:*

$$I_1=\int_{l^{\gamma}_{-\pi}(1,r)}\frac{\upsilon(|w|)\,dw}{z-w}=(1+ic)\lim_{\varepsilon\to 0+}\sum_{k=0}^{+\infty}\frac{(-1)^{k+1}}{z^{k+1}}A_k((1-\varepsilon)r,\upsilon)+o(\upsilon(r)),\tag{3.5}$$
$$I_2=z\int_{l^{\gamma}_{-\pi}(r,+\infty)}\frac{\upsilon(|w|)\,dw}{w(z-w)}=(1+ic)\lim_{\varepsilon\to 0+}\sum_{k=0}^{+\infty}\frac{(-1)^k}{z^{-k-1}}B_k((1+\varepsilon)r,\upsilon)+o(\upsilon(r)).\tag{3.6}$$

Proof. We have $w = te^{i(-\pi+\gamma(t))}$, $dw = (1+it\gamma'(t))e^{i(-\pi+\gamma(t))}\,dt$, $t\gamma'(t) = c+\alpha(t)$,

$$\begin{aligned} I_1 &= \int_1^r \frac{\upsilon(t)(1+it\gamma'(t))e^{i(-\pi+\gamma(t))}}{z(1-w/z)}\,dt \\ &= \lim_{\varepsilon\to 0+}\int_1^{(1-\varepsilon)r}\biggl(\frac{\upsilon(t)(1+ic+i\alpha(t))}{z}\,e^{i(-\pi+\gamma(t))}\sum_{k=0}^{+\infty}\Bigl(\frac{w}{z}\Bigr)^k\biggr)\,dt \\ &= (1+ic)\lim_{\varepsilon\to 0+}\sum_{k=0}^{+\infty}\frac{(-1)^{k+1}}{z^{k+1}}\int_1^{(1-\varepsilon)r}\upsilon(t)t^k e^{i(k+1)\gamma(t)}\,dt + i\int_1^r \frac{\upsilon(t)\alpha(t)e^{i(-\pi+\gamma(t))}}{z-w}\,dt \\ &= (1+ic)\lim_{\varepsilon\to 0+}\sum_{k=0}^{+\infty}\frac{(-1)^{k+1}}{z^{k+1}}A_k((1-\varepsilon)r,\upsilon) + o(\upsilon(r)), \qquad r\to+\infty, \end{aligned}$$

because, by Lemma 1, uniformly with respect to $\varphi \in [-\pi+\delta, \pi-\delta]$,

$$i\int_1^r \frac{\upsilon(t)\alpha(t)e^{i(-\pi+\gamma(t))}}{z-w}\,dt = i\int_{l^\gamma_{-\pi}(1,r)} \frac{\alpha(|w|)\upsilon(|w|)\,dw}{(1+it\gamma'(t))(z-w)} = o(\upsilon(r)), \qquad r\to+\infty.$$

Relation (3.6) is proved similarly. □

Recall that a set $E \subset \mathbb{R}_+$ is called an E_0-*set* if E is measurable and $\operatorname{mes}(E\cap[0,r]) = o(r)$, $r\to+\infty$. From [7, Lemmas 4 and 5], we obtain the following statement.

Lemma 4. *Suppose that $\upsilon \in L$, $0<\delta<1$, $\theta\in[-\pi,\pi)$, $f\in H_0(\upsilon)$. Then there exists an E_0-set E such that*

$$r\int_{\theta-\delta}^{\theta+\delta}\left|\frac{f'(re^{i\varphi})}{f(re^{i\varphi})}\right|d\varphi = O(\upsilon(r))\biggl(\delta+\delta\ln\Bigl(1+\frac{1}{\delta}\Bigr)\biggr), \qquad r\to+\infty, \quad r\notin E.$$

For the entire function f defined by (2.1), we denote by $f^\delta(z)$ the product $f^\delta(z) = \prod_{n=1}^{+\infty}(1-z/a'_n)$, in which $|a'_n| = |a_n|$ and $|\arg a'_n - \arg a_n| < \delta$.

Lemma 5. *Let $\upsilon\in L$, $f\in H_0(\upsilon)$, and let the following limit exist*:

$$\lim_{r\to\infty}\frac{n(r,0,f)}{\upsilon(r)} = \Delta \ge 0.$$

Then

$$\forall\,\varepsilon>0 \quad \forall\,\eta>0 \quad \exists\,\delta>0 \quad \exists\,E\subset\mathbb{C},\ \rho^*(E)<\eta\colon \quad \forall\,z\notin E \quad \bigl|\ln|f(z)| - \ln|f^\delta(z)|\bigr| < \varepsilon.$$

The proof of this lemma was given in [10].

Lemma 6. *Suppose that $\upsilon\in L$, $f\in H_0(\upsilon)$, the roots of f have an υ-density $\Delta^\gamma(\alpha,\beta)$ along the curves of regular rotation l^γ_φ, K is a closed set of ordinary curves of regular rotation l^γ_θ of f, and the function $f^\delta(z)$ is the same as above. Then*

$$\forall\,\varepsilon>0 \quad \forall\,z\in K \quad \exists\,\delta>0 \quad \exists\,r_0>0\colon \quad \forall\,r\ge r_0 \quad |\arg f(z) - \arg f^\delta(z)| < \varepsilon\upsilon(r).$$

Proof. Let us express f as the product of the following three functions:

$$f_1(z) = \prod_{|a_n|\le r/2}\Bigl(1-\frac{z}{a_n}\Bigr), \quad f_2(z) = \prod_{|a_n|\ge 2r}\Bigl(1-\frac{z}{a_n}\Bigr), \quad f_3(z) = \prod_{r/2<|a_n|<2r}\Bigl(1-\frac{z}{a_n}\Bigr).$$

Obviously,

$$|\arg f(z) - \arg f^\delta(z)|$$

$$\le |\arg f_1(z) - \arg f_1^\delta(z)| + |\arg f_2(z) - \arg f_2^\delta(z)| + |\arg f_3(z) - \arg f_3^\delta(z)|$$
$$\le |\ln f_1(z) - \ln f_1^\delta(z)| + |\ln f_2(z) - \ln f_2^\delta(z)| + |\arg f_3(z) - \arg f_3^\delta(z)|. \tag{3.7}$$

Since $n(r) \le \Delta\upsilon(r)$, $r \ge r_1$, $0 < \Delta < +\infty$, $\int_{2r}^{+\infty} \upsilon(t)/t\,dt \sim \upsilon(2r)/2r$, $r \to +\infty$, it follows that, for an arbitrary $\varepsilon > 0$, we can choose $0 < \delta < 1/4$ and $r_0 \ge r_1$ so that $8\Delta\delta < \varepsilon/3$, $\upsilon(2r) < 2\upsilon(r)$, $\int_{2r}^{+\infty} \upsilon(t)/t^2\,dt < \upsilon(2r)/r$ for $r \ge r_0$.

For $|\arg a_n' - \arg a_n| = |\arg \alpha_n| < \delta$, $|a_n| \le r/2$, and $|a_n| \ge 2r$, we have

$$\left|\frac{(a_n' - a_n)z}{a_n(a_n' - z)}\right| \le \frac{r|1 - e^{i\alpha_n}|}{||a_n| - r|} \le \frac{\delta r}{||a_n| - r|} \le 2\delta < \frac{1}{2}\,.$$

Using the inequality $|\ln(1-u)| \le 2|u|$ for $|u| \le 1/2$ (see [6, p. 87]), we obtain

$$|\ln f_1(z) - \ln f_1^\delta(z)| \le \sum_{|a_n|\le r/2} \left|\ln\left(1 - \frac{z}{a_n}\right) - \ln\left(1 - \frac{z}{a_n'}\right)\right|$$
$$= \sum_{|a_n|\le r/2} \left|\ln\left(1 - \frac{(a_n' - a_n)z}{a_n(a_n' - z)}\right)\right| \le 2 \sum_{|a_n|\le r/2} \left|\frac{(a_n' - a_n)z}{a_n(a_n' - z)}\right|$$
$$\le 4\delta n\left(\frac{r}{2}\right) \le 4\delta\Delta\upsilon(r) < \frac{\varepsilon}{3}\,\upsilon(r), \qquad r \ge r_0. \tag{3.8}$$

Since

$$\int_{2r}^{+\infty} \frac{dn(t)}{t} \le \int_{2r}^{+\infty} \frac{n(t)}{t^2}\,dt \le \Delta \int_{2r}^{+\infty} \frac{\upsilon(t)}{t^2}\,dt \le \frac{\Delta\upsilon(2r)}{r} \le 2\Delta\frac{\upsilon(r)}{r}\,,$$

we, similarly, have

$$|\ln f_2(z) - \ln f_2^\delta(z)| \le 2 \sum_{|a_n|\ge 2r} \left|\frac{(a_n' - a_n)z}{a_n(a_n' - z)}\right| \le 2\delta r \sum_{|a_n|\ge 2r} \frac{1}{|a_n| - r} = 2\delta r \int_{2r}^{+\infty} \frac{dn(t)}{t - r}$$
$$\le 4\delta r \int_{2r}^{+\infty} \frac{dn(t)}{t} \le 8\delta\Delta\upsilon(r) < \frac{\varepsilon}{3}\,\upsilon(r), \qquad r \ge r_0. \tag{3.9}$$

Using the same arguments as in [5, pp. 352–353], we obtain the estimate

$$|\arg f_3(z) - \arg f_3^\delta(z)| < \frac{\varepsilon}{3}\,\upsilon(r), \qquad z \in l_\theta^\gamma \in K, \quad r \ge r_0. \tag{3.10}$$

In view of (3.7), using (3.8)–(3.10), we conclude the proof of the assertion of Lemma 6. □

4. PROOF OF THE MAIN RESULTS

Without loss of generality, we assume that $\upsilon(r) = 0$ for $0 \le r \le 1$.

Proof of Theorem 1. Suppose that the roots (a_n) of the function $f \in H_0(\upsilon)$, $\upsilon \in L$, are on the curves of regular rotation $l_{-\pi}^\gamma$, $n(r) = \upsilon(r)(1 + \alpha(r))$, and $\alpha(r) \to 0$ as $r \to +\infty$. Then, for $z = re^{i(\varphi+\gamma(r))}$, $-\pi < \varphi < \pi$, we have (see [3])

$$\ln f(z) = \sum_{n=1}^{+\infty} \ln\left(1 - \frac{z}{a_n}\right) = \int_{l_{-\pi}^\gamma} \ln\left(1 - \frac{z}{w}\right) dn(|w|) = -z \int_{l_{-\pi}^\gamma} \frac{n(|w|)}{w(w - z)}\,dw$$
$$= \int_{l_{-\pi}^\gamma(1,r)} \frac{n(|w|)}{w}\,dw$$
$$+ \int_{l_{-\pi}^\gamma(1,r)} n(|w|)\left(\frac{z}{w(z - w)} - \frac{1}{w}\right) dw + \int_{l_{-\pi}^\gamma(r,+\infty)} \frac{n(|w|)}{w(z - w)}\,dw$$

$$= N^{\gamma}(r) + \int_{l^{\gamma}_{-\pi}(1,r)} \frac{n(|w|)}{z-w}\,dw + \int_{l^{\gamma}_{-\pi}(r,+\infty)} \frac{n(|w|)}{w(z-w)}\,dw$$
$$= N^{\gamma}(r) + I_1 + I_2 + J_1 + J_2, \tag{4.1}$$

where I_1, I_2, J_1, J_2 are the same as in Lemmas 1 and 3. By virtue of (3.1), (3.3), (3.5), we obtain

$$I_1 + J_1 = \lim_{\varepsilon\to 0+} \sum_{k=0}^{+\infty} \frac{(-1)^{k+1}}{z^{k+1}} \biggl\{ \frac{\upsilon((1-\varepsilon)r)(1-\varepsilon)^{k+1} r^{k+1} e^{i(k+1)\gamma((1-\varepsilon)r)}}{k+1}$$
$$- \frac{1}{k+1} A_k((1-\varepsilon)r, \widetilde{\upsilon}\,) - iA_k((1-\varepsilon)r, \alpha\cdot\upsilon) \biggr\} + o(\upsilon(r))$$
$$= -\upsilon(r) \lim_{\varepsilon\to 0+} \sum_{k=0}^{+\infty} \frac{(-1)^k}{k+1} (1-\varepsilon)^{k+1} \exp(i(k+1)(-\varphi + \gamma((1-\varepsilon)r) - \gamma(r)))$$
$$- \lim_{\varepsilon\to 0+} \sum_{k=0}^{+\infty} \frac{(-1)^{k+1}}{(k+1)z^{k+1}} A_k((1-\varepsilon)r, \widetilde{\upsilon}\,) - \frac{i}{1+ic} \int_{l^{\gamma}_{-\pi}(1,r)} \frac{\alpha(|w|)\upsilon(|w|)}{z-w}\,dw + o(\upsilon(r))$$
$$= -\upsilon(r) \lim_{\varepsilon\to 0+} \ln(1 + (1-\varepsilon)\exp(i(-\varphi + \gamma((1-\varepsilon)r) - \gamma(r)))) - \lim_{\varepsilon\to 0+} \Sigma_{1,\varepsilon} + o(\upsilon(r))$$
$$= -\ln(1+e^{-i\varphi})\upsilon(r) + o(\upsilon(r)), \qquad r\to+\infty, \tag{4.2}$$

because, by Lemma 3 from [7],

$$|\Sigma_{1,\varepsilon}| = \biggl| \sum_{k=0}^{+\infty} \frac{(-1)^{k+1}}{(k+1)z^{k+1}} A_k((1-\varepsilon)r, \widetilde{\upsilon}\,) \biggr|$$
$$\le \sum_{k=0}^{+\infty} \frac{1}{(k+1)r^{k+1}} \int_1^r \widetilde{\upsilon}(t) t^k\,dt = o(\upsilon(r)), \qquad r\to+\infty.$$

Similarly, from (3.2), (3.4), (3.6), and [7, Lemmas 3], we obtain

$$I_2 + J_2 = \ln(1+e^{i\varphi})\upsilon(r) + o(\upsilon(r)), \qquad r\to+\infty. \tag{4.3}$$

Using (4.1)–(4.3) for $z = re^{i(\varphi+\gamma(r))}$, $-\pi<\varphi<\pi$, we can write

$$\ln f(z) = N^{\gamma}(r) + (\ln(1+e^{i\varphi}) - \ln(1-e^{-i\varphi}))\upsilon(r) + o(\upsilon(r))$$
$$= N^{\gamma}(r) + i\varphi\upsilon(r) + o(\upsilon(r)), \qquad r\to+\infty; \tag{4.4}$$

here the latter relation holds uniformly with respect to $\varphi\in[-\pi+\delta, \pi-\delta]$, $0<\delta<1$.

If the roots $f\in H_0(\upsilon)$ are on the curves of regular rotation l^{γ}_{ψ}, $-\pi<\psi<\pi$, then, by turning the plane clockwise by the angle $(\pi+\psi)$, i.e., replacing in (4.4) the value of φ by $\varphi-\psi-\pi$, we see that, for an arbitrary $\delta\in(0,1)$, the following relation holds uniformly with respect to $\varphi\in[\psi+\delta, \psi+2\pi-\delta]$:

$$\ln f(re^{i(\varphi+\gamma(r))}) = N^{\gamma}(r) + ih(\varphi;\psi)\upsilon(r) + o(\upsilon(r)), \qquad r\to+\infty. \tag{4.5}$$

Now let the roots of $f\in H_0(\upsilon)$ be placed on a finite system of curves of regular rotation $\Gamma_m = \bigcup_{j=1}^m l^{\gamma}_{\theta_j}$, $-\pi\le\theta_1<\theta_2<\cdots<\theta_m<\pi=\theta_{m+1}$, and $n(r;\theta_j) = \Delta_j\upsilon(r) + o(\upsilon(r))$, $r\to+\infty$. Let us express f as the product $f = f_1\cdots f_m$, where f_j is an entire function with roots on the curves of regular rotation $l^{\gamma}_{\theta_j}$, $j=1,\dots,m$. Then, for $z\in D(\Gamma_m) = \mathbb{C}\setminus\Gamma_m$, we have

$$\ln f(z) = \ln f_1(z) + \cdots + \ln f_m(z)$$

and, using (4.5), for $z = re^{i(\varphi+\gamma(r))}$, $\varphi\ne\theta_j$, $j=1,\dots,m$, we obtain

$$\ln f(z) = N^{\gamma}(r) + i\sum_{j=1}^m \Delta_j \widehat{h}(\varphi;\theta_j)\upsilon(r) + o(\upsilon(r)), \qquad r\to+\infty.$$

Consider the case of an arbitrary placement of the roots of $f \in H_0(\upsilon)$. We will employ the methods used in a similar situation in [5, pp. 348–349] and [6, pp. 162–163]. Let us construct the integral sum

$$S_m(\varphi) = \sum_{j=1}^{m} \Delta_j \widehat{h}(\varphi; \psi_j), \qquad \Delta_j = \Delta(\psi_{j+1}) - \Delta(\psi_j),$$

where $-\pi \le \psi_1 < \psi_2 < \cdots < \psi_m < \psi_{m+1} = \pi$. For any $\varepsilon > 0$, we can choose a $\delta > 0$, so that, for $\max_{1\le j\le m} |\psi_{j+1} - \psi_j| < \delta$, the following inequality holds:

$$|H_f^\gamma(\varphi) - S_m(\varphi)| < \frac{\varepsilon}{6}\,. \tag{4.6}$$

Now let us take numbers a_n' so that $|a_n'| = |a_n|$, and if $a_k \in l_\psi^\gamma$, $\psi_j \le \psi_{j+1}$, then $a_k' \in l_{\psi_j}^\gamma$, $j = 1, 2, \dots, m$, and we construct the function $f^\delta(z)$. Applying Lemmas 5 and 6, we see that, for any $\varepsilon > 0$, $\eta > 0$, and a sufficiently small $\delta > 0$, the inequality

$$\big|\ln|f(z)| - \ln|f^\delta(z)|\big| < \frac{\varepsilon}{4}\,\upsilon(r) \tag{4.7}$$

holds for $z \notin E$, $\rho^*(E) < \eta/2$, and the inequality

$$|\arg f(z) - \arg f^\delta(z)| < \frac{\varepsilon}{6}\,\upsilon(r) \tag{4.8}$$

holds for $z \in l_\varphi^\gamma$, $r \ge r_1$, where l_φ^γ are the ordinary curves of regular rotation of the function f. The roots of $f^\delta(z)$ lie on a finite system of curves of regular rotation. $\Gamma_m = \bigcup_{j=1}^m l_{\psi_j}^\gamma$, $n(r; \psi_j) = (1 + o(1))\Delta_j \upsilon(r)$, $r \to +\infty$, and, as proved above, for arbitrary $\varepsilon > 0$, $\sigma > 0$, for $r \ge r_2$ and $\sigma \le \varphi - \psi_j \le 2\pi - \sigma$, $j = 1, 2, \dots, m$, the following inequalities are valid ($z \in l_\varphi^\gamma$):

$$|\ln|f^\delta(z)| - N^\gamma(z)| < \frac{\varepsilon}{4}\,\upsilon(r), \qquad |\arg f^\delta(z) - S_m(\varphi)| < \frac{\varepsilon}{6}\,\upsilon(r).$$

Combining this with (4.6)–(4.8), we see that, for $z \notin E$, $z \in l_\varphi^\gamma$, $\sigma \le \varphi - \psi_j \le 2\pi - \sigma$, $j = 1, 2, \dots, m$,

$$|\ln|f(z)| - N^\gamma(z)| < \frac{\varepsilon}{2}\,\upsilon(r), \quad |\arg f(z) - H_f^\gamma(\varphi)\upsilon(r)| < \frac{\varepsilon}{2}\,\upsilon(r), \quad r \ge r_0 = \max\{r_1, r_2\},$$

i.e.,

$$|\ln f(z) - N^\gamma(z) - iH_f^\gamma(\varphi)\upsilon(r)| < \varepsilon\upsilon(r). \tag{4.9}$$

Performing a second partition of the interval $[-\pi, \pi)$ by points ψ_j', $j = 1, 2, \dots, m$, so that the inequalities $\sigma \le \theta - \psi_j \le 2\pi - \sigma$ and $\sigma \le \theta - \psi_j' \le 2\pi - \sigma$ cover the whole interval $[-\pi, \pi]$, we can see that (4.9) hold for $z = re^{i(\varphi+\gamma(r))} \notin E_1$, $\rho^*(E_1) < \eta$. Further, just as in [6, p. 133], we construct a C_0-set outside which (2.2) holds. Theorem 1 is proved. □

Proof of Theorem 2. Suppose that $r \notin \Omega = \{|a_n|\colon n \in \mathbb{N}\}$, the a_n are the roots of $f \in H_0(\upsilon)$, $D^\gamma(r; \alpha, \beta)$ is a curvilinear sector, and

$$-\pi \le \theta_1 < \cdots < \theta_{k_0-1} < \alpha < \theta_{k_0} < \cdots < \theta_{l_0} < \beta < \theta_{l_0+1} < \cdots < \theta_m < \pi.$$

Denote by

$$\partial D^\gamma(r; \alpha, \beta) = l_\alpha^\gamma(1, r) \cup K(r; \alpha, \beta) \cup (l_\beta^\gamma(1, r))^{-1} \cup (K(1; \alpha, \beta))^{-1}$$

the positive orientation of the boundary $D^\gamma(r; \alpha, \beta)$, where

$$K(t; \alpha, \beta) = \{z : |z| = t,\ \alpha + \gamma(t) \le \arg z \le \beta + \gamma(t)\}.$$

In view of (2.3) and the condition that the roots of f lie on the the curves of regular rotation $l_{\theta_j}^\gamma$, $j = 1, \dots, m$, by the residue theorem, we have

$$2\pi i n^\gamma(r; \alpha, \beta) = \int_{\partial D^\gamma(r;\alpha,\beta)} \frac{f'(z)}{f(z)}\,dz = \bigg(\int_{l_\alpha^\gamma(1,r)} + \int_{K(r;\alpha,\beta)} - \int_{l_\beta^\gamma(1,r)} - \int_{K(1;\alpha,\beta)}\bigg) \frac{f'(z)}{f(z)}\,dz$$

$$\begin{aligned}
&= \int_1^r \frac{f'(te^{i(\alpha+\gamma(t))})}{f(te^{i(\alpha+\gamma(t))})}(1+it\gamma'(t))e^{i(\alpha+\gamma(t))}\,dt \\
&\quad - \int_1^r \frac{f'(te^{i(\beta+\gamma(t))})}{f(te^{i(\beta+\gamma(t))})}(1+it\gamma'(t))e^{i(\beta+\gamma(t))}\,dt \\
&\quad + \int_\alpha^\beta \frac{f'(re^{i(\varphi+\gamma(r))})}{f(re^{i(\varphi+\gamma(r))})}\,ire^{i(\varphi+\gamma(r))}\,d\varphi - \int_\alpha^\beta \frac{f'(e^{i(\varphi+\gamma(1))})}{f(e^{i(\varphi+\gamma(1))})}\,ie^{i(\varphi+\gamma(1))}\,d\varphi \\
&= \ln f(re^{i(\alpha+\gamma(r))}) - \ln f(re^{i(\beta+\gamma(r))}) + J(r;\alpha,\beta) + C \\
&= i(G(\alpha)-G(\beta))\upsilon(r) + J(r;\alpha,\beta) + C + o(\upsilon(r)), \qquad r\to+\infty, \qquad (4.10)
\end{aligned}$$

where

$$C = -\ln f(e^{i(\alpha+\gamma(1))}) + \ln f(e^{i(\beta+\gamma(1))}) - i\int_\alpha^\beta \frac{f'(e^{i(\varphi+\gamma(1))})}{f(e^{i(\varphi+\gamma(1))})}\,e^{i(\varphi+\gamma(1))}\,d\varphi.$$

Let

$$0<\delta<\min\left\{\frac{\theta_{k_0}-\alpha}{2},\,\frac{\beta-\theta_{l_0}}{2},\,\frac{\theta_{j+1}-\theta_j}{2}\right\}, \qquad j=k_0,\dots,l_0-1.$$

Then, by (2.3), we have

$$\begin{aligned}
&J(r;\alpha,\beta) \\
&= \left(\int_\alpha^{\theta_{k_0}-\delta} + \sum_{j=k_0}^{l_0-1}\int_{\theta_j+\delta}^{\theta_{j+1}-\delta} + \sum_{j=k_0}^{l_0}\int_{\theta_j-\delta}^{\theta_j+\delta} + \int_{\theta_{l_0}+\delta}^{\beta}\right)\frac{f'(re^{i(\varphi+\gamma(r))})}{f(re^{i(\varphi+\gamma(r))})}\,ire^{i(\varphi+\gamma(r))}\,d\varphi \\
&= i(G(\theta_{k_0}-\delta)-G(\alpha))\upsilon(r) + i\sum_{j=k_0}^{l_0-1}(G(\theta_{j+1}-\delta)-G(\theta_j+\delta))\upsilon(r) \\
&\quad + i(G(\beta)-G(\theta_{l_0}+\delta))\upsilon(r) + \Sigma_\delta + o(\upsilon(r)), \qquad r\to+\infty, \qquad (4.11)
\end{aligned}$$

where

$$\Sigma_\delta = \sum_{j=k_0}^{l_0}\int_{\theta_j-\delta}^{\theta_j+\delta}\frac{f'(re^{i(\varphi+\gamma(r))})}{f(re^{i(\varphi+\gamma(r))})}\,ire^{i(\varphi+\gamma(r))}\,d\varphi.$$

By Lemma 4, there exists an E_0-set of E_j such that

$$\begin{aligned}
&\left|\int_{\theta_j-\delta}^{\theta_j+\delta}\frac{f'(re^{i(\varphi+\gamma(r))})}{f(re^{i(\varphi+\gamma(r))})}\,ire^{i(\varphi+\gamma(r))}\,d\varphi\right| \le r\int_{\theta_j-\delta}^{\theta_j+\delta}\left|\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right|d\theta \\
&\quad = O(\upsilon(r))\left(\delta+\delta\ln\left(1+\frac{1}{\delta}\right)\right), \qquad r\to+\infty, \quad r\notin E_j, \quad j=k_0,\dots,l_0,
\end{aligned}$$

and hence

$$\Sigma_\delta = O(\upsilon(r))\left(\delta+\delta\ln\left(1+\frac{1}{\delta}\right)\right), \qquad r\to+\infty, \quad r\notin E=\bigcup_{j=k_0}^{l_0}E_j. \qquad (4.12)$$

Letting δ tend to zero, from (4.10)–(4.12), we obtain

$$n^\gamma(r;\alpha,\beta) = \frac{1}{2\pi}\sum_{j=k_0}^{l_0}(G(\theta_j-0)-G(\theta_j+0))\upsilon(r) + o(\upsilon(r)), \qquad r\to+\infty, \quad r\notin E,$$

where E is some E_0-set. If $r\in E$, then it follows from the definition of E that there exist r', r'' such that $r/2<r'<r<r''<2r$ and $r'\notin E$, $r''\notin E$. Since

$$\frac{n^\gamma(r';\alpha,\beta)}{\upsilon(r')}\frac{\upsilon(r')}{\upsilon(r)} \le \frac{n^\gamma(r;\alpha,\beta)}{\upsilon(r)} \le \frac{n^\gamma(r'';\alpha,\beta)}{\upsilon(r'')}\frac{\upsilon(r'')}{\upsilon(r)},$$

$\upsilon(r') \sim \upsilon(r'') \sim \upsilon(r)$ as $r \to +\infty$ and $G(\theta_j - 0) - G(\theta_j + 0) = 2\pi\Delta_j$, it follows that

$$\Delta^\gamma(\alpha, \beta) = \lim_{r\to+\infty} \frac{n^\gamma(r; \alpha, \beta)}{\upsilon(r)} = \sum_{j=k_0}^{l_0} \Delta_j,$$

which proves Theorem 2. □

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REFERENCES

1. A. Macintyre, "Laplace's transformation and integral functions," Proc. London Math. Soc. (2) **45** (2), 1–20 (1938).
2. P. Kennedy, "A class of integral functions bounded on certain curves," Proc. London Math. Soc. **3** (6), 518–547 (1956).
3. S. K. Balashov, "On entire functions of finite order with zeros on curves of regular rotation," Math. USSR-Izv. **7** (3), 601–627 (1973).
4. A. I. Kheifits, "Analogue of the Valiron–Titchmarsh theorem for entire functions with roots on a logarithmic spiral," Soviet Math. (Iz. VUZ) **24** (12), 92–94 (1980).
5. S. K. Balashov, "On entire functions of completely regular growth along curves of regular rotation," Math. USSR-Izv. **10** (2), 321–338 (1976).
6. B. Ya. Levin, *Distribution of Zeros of Entire Functions* (Gostekhizdat, Moscow, 1956) [in Russian].
7. N. V. Zabolotskii, "Strongly regular growth of entire functions of order zero," Math. Notes **63** (2), 172–182 (1998).
8. M. V. Zabolotskii, "An example of entire function of strongly regular growth," Mat. Stud. **13** (2), 145–148 (2000).
9. A. A. Gol'dberg and I. V. Ostrovskii, *The distribution of Values of Meromorphic Functions* (Nauka, Moscow, 1970) [in Russian].
10. M. Zabolotskyi and Yu. Basiuk, "Proximity of the entire functions of zero order with υ-density of zeros," Visnyk of the Lviv Univ. Series Mech. Math. **84**, 80–86 (2017), `http:// nbuv.gov.ua/ UJRN/ VLNU_Mech_mat_2017_84_13`.