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Archer–bow–arrow behaviour in the vertical plane

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Theoretical and experimental results of research on the problem of archer, bow and arrow behaviour in the vertical plane are presented. The aim of the research is to develop a method of computer simulation of static and dynamic interactions between the archer, bow and arrow system in order to provide archery with practical recommendations. A model of an archer's body is presented as a mechanical system composed of a few solid bodies, which are connected to each other and to the ground with viscoelastic elements. Mechanical and mathematical model of bow and arrow geometry in vertical plane in braced and drawn situations is investigated. An asymmetrical scheme, rigid beams, concentrated elastic elements and elastic string are the main features of the model. Numerical results of computer simulation of archer, bow and arrow interactions are presented in graphical form, which makes the methods easy to use by sportsmen and coaches.

Key words: archery, biomechanics, mathematical modelling, computer simulation

1. Introduction

During a rapid motion an arrow is in common motion space with a bow and an archer. The main motion in the vertical plane is caused by the bow–grip–limbs–string system. Simultaneously the arrow through the string nock point is involved by the whole system in deflection out of the vertical plane. This rapid motion causes arrow deflection in the lateral plane.

Rigorous three-dimensional analysis of the space system is very complicated and with the other assumptions is not essential. The main part of potential energy, stored in bow–limbs, is transformed to kinetic energy of the longitudinal motion of the arrow and the string common motion. Some negligible part of energy is transformed and deflects the motion of the arrow. Although the arrow is in a space motion of the whole system, the problem can be idealised and reduced to two separate systems. The first one is in the vertical plane, and the second is in the lateral plane.

The first step in the understanding of an arrow behaviour in internal ballistics was related to the explanation of the phenomenon that for almost two centuries has been known as the archer's paradox. The investigation of the archer's paradox was carried out by means of a high-speed spark photography which allowed us to secure direct evidence of what an arrow does as it leaves the bow [3]. The archer's paradox is the fact that an arrow does not fly at a target along the line represented by its axis. The forces acting on the arrow during its release do not coincide sufficiently with this axis. The string force is exerted on the arrow in the bow plane. In a starting position, the arrow does not lie in this plane, and its axis inclines towards it at an angle of a few degrees. Even in the case the nock and head points of the arrow do lie in the bow plane, the longitudinal arrow axis of the string force line does not coincide enough with the bow plane because the initial shape of the arrow axis is not straight enough. So, the line of string force does not coincide with the line of cross-section centres of the arrow. The released string pushes the arrow's nock point into the bow plane. Therefore the arrow moves forwards and slightly turns decreasing the angle with this plane. The impulse normal to the axis of the arrow caused by the release of the fingers from the string, as well as the column-like force of the string push the arrow during its acceleration motion resulting in a significant bending of the arrow shaft as it transits the bow. All these factors allow the arrow to undulate around the bow handle and to follow a straight course towards its target without striking the bow handle. Mechanical and mathematical model of the lateral motion was investigated by PĘKALSKI [6] and improved by KOOI [4] and ZANEVSKYY [8], [9].

We started analytical modelling of the system behaviour in the vertical plane using pseudo-static [2], [10] and dynamic approaches [11]. A dynamic analysis of the traditional asymmetrical bow has been carried out in Japan [5], but such a bow is very different from a modern sports bow and in its design human parameters have not been taken into account. Hence, there are no analytical models, which make the studies of the archer, bow and arrow interaction in the vertical plane possible and which improve computational methods for adjusting the parameters of the system.

The aim of the research was to develop a method of mathematical modelling and computer simulation of static and dynamic interactions in the archer–bow–arrow system in order to provide archery with practical recommendations.

In the paper, the use was made of biomechanical methods (the model of human body as a system of mechanical oscillators); theoretical mechanics and mathematics (Lagrange's equations of the second order; the principle of d'Alembert; the Cauchy problem, method of iterations and the Runge–Kutta method); computational methods (MathCAD and Mathematica packets); experimental methods (high-speed video analysis; natural mechanical modelling).

2. Archer's body modelling

In theoretical modelling, a human body is presented as a mechanical system composed of a few solid bodies, which are connected to each other with viscoelastic elements [12]. A model structure is built taking into consideration the aim of a research, body position and the character of its interaction with the surroundings. Sport archers string a bow during the time of a common motion of a string and an arrow with fixed hand trying to keep a steady position of a body. The motor program of shooting is initiated before the clicker (a signal to release) can be heard [1].

Bow interacts with archer in the point of contact (*H*), i.e., the point at which archer's hand is in contact with bow handle (figure 1). Because of a short time of string and arrow common motion (0.01–0.02 s) and relatively deep contact along the inner side of the thumb muscle, the point of contact (*H*) can be assumed to be a pivot point. Because of a small displacement of the bow handle, we can reduce the motion to orthogonal directions, which are suitable to describe the motion of bow hand in the vertical plane; they are as follows: the longitudinal $H\xi$-axis (along the upper extremity) and the $H\eta$-axis perpendicular to it.

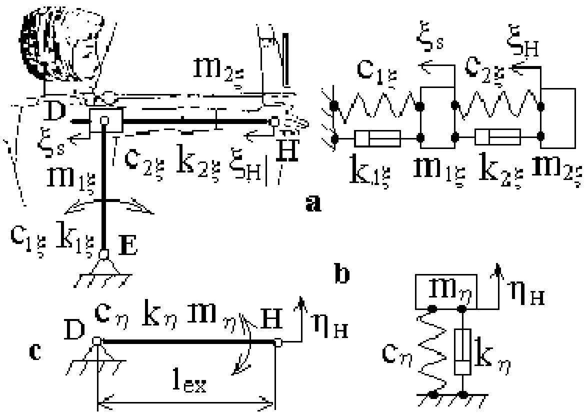

Fig. 1. A model of the archer's body: a – longitudinal direction, b – perpendicular direction

The next step to obtain a suitable model is a kinematical linkage of the group with lower pairs (*E-D-H*). In the longitudinal direction, the number of the degrees of freedom equals two, i.e., the displacement of the body relative to a ground (rotation motion of the point *D* relative to the point *E*) and the displacement of the upper extremity (contacting a bow handle) relative to the body (*HD*). In the perpendicular direction, we deal with one degree of freedom, because elasticity of the shoulder girdle (*D*) is significantly greater than its elasticity relative to the body in the vertical direction (*DE*).

Thus we can model an archer's body in the longitudinal direction based on two particles and two viscoelastic elements. The first particle with the mass $m_{1\xi}$ models the archer's body (except the upper extremity that is in contact with a bow handle) and has a virtual coordinate ξ_s. The first viscoelastic element describes interaction in the body–ground system and is represented by the coefficients of stiffness $c_{1\xi}$ and the viscosity $k_{1\xi}$. The second one, i.e., an upper extremity (of the mass $m_{2\xi}$), has a virtual coordinate ξ_H because of its common motion with a handle's contact point *H*. The corresponding viscoelastic element with the coefficients $c_{2\xi}$ and $k_{2\xi}$ describes the interaction between the upper extremity and the remaining part of archer's body.

The model of motion in the perpendicular direction includes one particle (of the mass m_η) with a virtual coordinate η_H and viscoelastic element with the coefficients c_η and k_η.

Taking account of the features given above, we can derive the expressions representing the kinetic energy of the archer's body model (see figure 1):

$$T_{\text{archer}} = \frac{1}{2}(m_{1\xi}\xi_s'^2 + m_{2\xi}\xi_H'^2 + m_\eta \eta_H'^2);$$

potential energy:

$$P_{\text{archer}} = \frac{1}{2}[c_{1\xi}\xi_s^2 + c_{2\xi}(\xi_s - \xi_H)^2 + c_\eta \eta_H^2]; \tag{1}$$

dissipation function:

$$\Phi_{\text{archer}} = \frac{1}{2}[k_{1\xi}\xi_s'^2 + k_{2\xi}(\xi_s' - \xi_H')^2 + k_\eta \eta_H'^2].$$

Prime symbol expresses a derivation with time, i.e., $(') \equiv d/dt$.

3. Bow and arrow modelling

The basic assumption in the modelling of a handle, limbs and an arrow is their motion as rigid bodies in a general (vertical) plane. A handle, a stabiliser and a sign could be modelled as a one rigid body whose kinetic energy (figure 2) is expressed as follows:

$$T_{\text{handle}} = \frac{1}{2} \int_{m_h} (\xi_h'^2 + \eta_h'^2)\, dm_h\,, \tag{2}$$

where $\xi_h = \xi_H + y + x\kappa$; $\eta_h = \eta_H + x - y\kappa$; κ is an angular displacement relative to the point *H*; *Hxy* is a rectangular system of coordinate fixed to the handle. Inserting the two last expressions into (2) we obtain:

$$T_{\text{handle}} = \frac{1}{2}[m_H(\xi_H'^2 + \eta_H'^2) + I_H\kappa'^2 + 2m_H\kappa'(\xi_H' x_{CH} - \eta_H' y_{CH})]\,, \tag{3}$$

where m_H is mass; x_{CH}, y_{CH} are the coordinates of a centre of gravity; I_H is the moment of inertia relative to the point *H*.

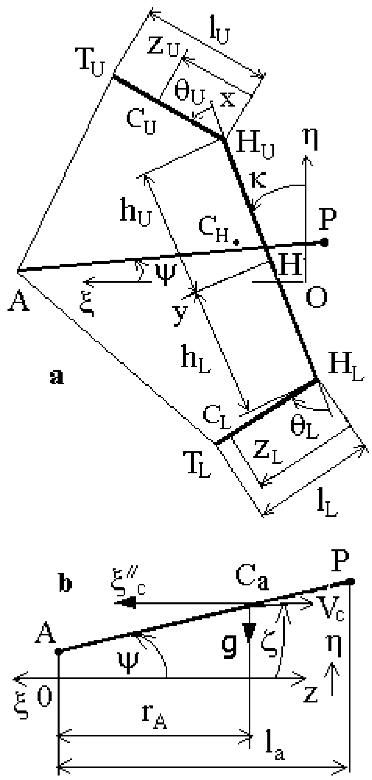

Fig. 2. Dynamic schematic models of a bow (a) and an arrow (b)

Kinetic energy of limbs is represented by:

$$T_{U/L} = \frac{1}{2}\int_0^{l_{U/L}} \mu(z_{U/L})\,(\xi'^2_{U/L} + \eta'^2_{U/L})\,dz_{U/L},$$

where $l_{U/L}$ is the length of a limb; μ is the distributed mass; z is the co-ordinate fixed to a limb. The upper and lower limbs are marked with the corresponding subscripts U and L. Inserting the expressions for displacements

$$\xi_{U/L} = \xi_H \pm \kappa h_{U/L} + z_{U/L}\sin(\theta_{U/L} \pm \kappa)\,; \quad \eta_{U/L} = \eta_H \pm z_{U/L}\cos(\theta_{U/L} \pm \kappa)$$

into the last equation for the energy $T_{U/L}$ we arrive at:

$$T_{\text{limbs}} = \frac{1}{2}\left\{\begin{array}{l} (m_U + m_L)(\xi'^2_H + \eta'^2_H) + (m_U h_U^2 + m_L h_L^2)\kappa'^2 + I_U(\theta'_U + \kappa')^2 + I_L(\theta'_L - \kappa')^2 \\ + 2(m_U r_U - m_L r_L)\xi'_H\kappa' + 2\kappa'\begin{bmatrix} m_U r_U h_U(\theta'_U + \kappa')\cos(\theta_U + \kappa) \\ - m_L r_L h_L(\theta'_L - \kappa')\cos(\theta_L - \kappa) \end{bmatrix} \\ + 2\xi'_H\begin{bmatrix} m_U r_U(\theta'_U + \kappa')\cos(\theta_U + \kappa) \\ + m_L r_L(\theta'_L - \kappa')\cos(\theta_L - \kappa) \end{bmatrix} - 2\eta'_H\begin{bmatrix} m_U r_U(\theta'_U + \kappa')\sin(\theta_U + \kappa) \\ - m_L r_L(\theta'_L - \kappa')\sin(\theta_L - \kappa) \end{bmatrix} \end{array}\right\}, \quad (4)$$

where $m_{U/L}$ is the mass of limbs; $I_{U/L}$ is the moment of inertia of limbs relative to ends of the handle (the points $H_{U/L}$); $r_{U/L}$ is the distance from the end of the handle to the limb's centre of mass.

The equation for potential energy of limbs may be expressed as follows (see figure 2a):

$$P_{U/L} = \frac{1}{2}c_{U/L}(\theta_{U/L} + \varphi_{U/L})^2\,, \quad (5)$$

where c_U and c_L stand for the stiffness of limbs.
Expression for the kinetic energy of a string is divided into three parts representing three parts of string mass pinned to the nock points of the limbs and the arrow:

$$T_{\text{string}} = \frac{1}{2}\frac{m_s}{3}\left[\xi'^2_A + \eta'^2_A + \frac{2s_U^*}{s^*}(\xi'^2_{TU} + \eta'^2_{TU}) + \frac{2s_L^*}{s^*}(\xi'^2_{TL} + \eta'^2_{TL})\right], \quad (6)$$

where m_s is the mass of a string; $s^*_{U/L}$ is the length of the string branches in unstrung bow $s^* = s^*_U + s^*_L$.

The expression for the potential energy of string branches is:

$$P_{s(U/L)} = \frac{f\,(s_{U/L} - s^{*}_{U/L})^2}{2s^{*}_{U/L}}, \tag{7}$$

where f is the distributed stiffness of the string; $s_{U/L}$ stands for the length of the string branches in strung bow.

The expression for the kinetic energy of the arrow (see figure 2b) is given by:

$$T_{\text{arrow}} = \frac{1}{2}\int_0^{l_a} \mu_a(z_a)(\xi_a'^2 + \eta_a'^2)dz_a + \frac{1}{2}m_P[\xi_a'^2 + \eta_a'^2]_{z_a = l_a}, \tag{8}$$

where l_a is the length of the arrow; μ_a is the distributed mass; z_a is the coordinate fixed to its longitudinal axis; m_P is the mass of the arrow's head.

The interaction between bow and arrow is described according the actual model only through the contact at the nock point. An initial position of the arrow relative to the bow in its main (vertical) plane is determined by the remains that holds an arrow's head. The remains are fixed to the handle and have the ability to turn and disappear just at the moment an arrow starts to move. Due to a small size and mass, the remains do not accumulate energy, therefore their interaction with an arrow is not taken into account within the framework of the model.

The results of a high-speed video analysis show that an arrow motion in the vertical plane can be assumed to be a rigid shift [7], 12]. Inserting $\xi_a = \xi_A$ and $\eta_a = \eta_A + z_a\psi$ into the energy equation (8) we obtain:

$$T_{\text{arrow}} = \frac{1}{2}m_a(\xi_A'^2 + \eta_A'^2 + 2r_A\eta_A'\psi') + I_A\psi'^2, \tag{9}$$

where m_a is the mass of the arrow and r_A stands for the distance from the tail to the arrow's centre of mass.

The expression for the potential energy of the arrow is calculated as a work of inertial forces (according the principle of d'Alembert) on the virtual longitudinal displacements ($\psi << 1$):

$$P_{\text{arrow}} = m_a r_A\left[\frac{1}{2}\xi_A''(\psi^2 - \psi_0^2) + g(\psi - \psi_0)\right], \tag{10}$$

where ψ_0 is an initial attitude angle of the arrow.

4. Dynamic behaviour

After some mathematical transformations in (1), (3)–(7), (9), (10), we get the expressions for the whole kinetic and potential energy of the archer–bow–arrow system during their common motion between string release and arrow shooting:

$$T=\frac{1}{2}\left\{\begin{array}{l}(m_{2\xi}+m_H+m_U+m_L)\,\xi_H'^2+(m_\eta+m_H+m_U+m_L)\eta_H'^2\\+(I_H+m_U h_U^2+m_L h_L^2)\kappa'^2+2m_H\kappa'(\xi_H' x_{CH}-\eta_H' y_{CH})\\+I_U(\theta_U'+\kappa')^2+I_L(\theta_L'-\kappa')^2+2(m_U r_U-m_L r_L)\xi_H'\kappa'\\+m_A(\xi_A'^2+\eta_A'^2)+2m_a r_A\eta_A'\psi'+I_A\psi'^2+m_{1\xi}\xi_s'^2\\+2\kappa'[m_U r_U h_U(\theta_U'+\kappa')b_1-m_L r_L h_L(\theta_L'-\kappa')b_3]\\+2\xi_H'[m_U r_U(\theta_U'+\kappa')b_1+m_L r_L(\theta_L'-\kappa')b_3]\\-2\eta_H'[m_U r_U(\theta_U'+\kappa')b_2-m_L r_L(\theta_L'-\kappa')b_4]\end{array}\right\},$$

$$P=\frac{1}{2}\left[\begin{array}{l}c_{1\xi}\xi_s^2+c_{2\xi}(\xi_s-\xi_H)^2+c_\eta\eta_H^2+c_U(\theta_U+\varphi_U)^2+c_L(\theta_L+\varphi_L)^2\\+\frac{f}{s_U^*}(s_U-s_U^*)^2+\frac{f}{s_L^*}(s_L-s_L^*)^2+m_a r_A\left[\xi_A''(\psi^2-\psi_0^2)+2g(\psi-\psi_0)\right]\end{array}\right], \quad (11)$$

where $b_1=\cos(\theta_U+\kappa)$; $b_2=\sin(\theta_U+\kappa)$; $b_3=\cos(\theta_L-\kappa)$; $b_4=\sin(\theta_L-\kappa)$; $m_A=\frac{1}{3}m_s+m_a$. Other two parts of the string mass have been taken into account with mass-inertial characteristics of limbs as pinned to the nock points.

Solving the dynamic problem, we do not consider gravity forces acting on the bow. But we take into consideration arrow weight because its force moment acting on the arrow has the same value as the moment of inertial forces.

Inserting expressions (1) and (11) into the Lagrange equations of the second order

$$\frac{d}{dt}\left(\frac{\partial T}{\partial q_i'}\right)-\frac{\partial T}{\partial q_i}+\frac{\partial P}{\partial q_i}+\frac{\partial \Phi}{\partial q_i'}=F_i,$$

we get a system of differential equations of the second order relative to generated co-ordinates $q_i\equiv\xi_s,\xi_H,\eta_H,\kappa,\theta_U,\theta_L,\xi_A,\eta_A,\psi$:

$$m_{1\xi}\xi_s''+c_{1\xi}\xi_s+c_{2\xi}(\xi_s-\xi_H)+k_{1\xi}\xi_s'+k_{2\xi}(\xi_s'-\xi_H')=F_{1\xi};$$

$$
\begin{gathered}
(m_H + m_U + m_L + m_{2\xi})\xi_H'' + c_{2\xi}(\xi_H - \xi_s) + k_{2\xi}(\xi_H' - \xi_s') \\
+ m_U r_U [b_1(\theta_U'' + \kappa'') - b_2(\theta_U' + \kappa')^2] + m_L r_L [b_3(\theta_L'' - \kappa'') - b_4(\theta_L' - \kappa')^2] \\
+ (m_H x_{CH} + m_U h_U - m_L h_L)\kappa'' + e_U S_{U\xi} + e_L S_{L\xi} = F_{2\xi}; \\[6pt]
m_U r_U [b_2(\theta_U'' + \kappa'') + b_1(\theta_U' + \kappa')^2] + m_L r_L [b_4(\theta_L'' - \kappa'') + b_3(\theta_L' - \kappa')^2] \\
+ (m_H + m_U + m_L + m_\eta)\eta_H'' + c_\eta \eta_H + k_\eta \eta_H' - m_H y_{CH}\kappa'' - e_U S_{U\eta} + e_L S_{L\eta} = F_\eta; \\
(I_H + I_U + I_L + m_U h_U^2 + m_L h_L^2)\kappa'' + I_U \theta_U'' - I_L \theta_L'' \\
+ m_U r_U h_U [b_1(\theta_U'' + 2\kappa'') - b_2(\theta_U' + \kappa')^2] - m_L r_L h_L \left[b_3(\theta_L'' - 2\kappa'') - b_4(\theta_L' - \kappa')^2\right] \\
+ [m_H x_{CH} + m_U(h_U + b_1 r_U) - m_L(h_L + b_3 r_L)]\xi_H'' - (m_H y_{CH} + m_U r_U b_2 + m_L r_L b_4)\eta_H'' \\
+ e_U [S_{U\xi}(b_1 l_U + h_U) - S_{U\eta} b_2 l_U] - e_L [S_{L\xi}(b_3 l_L + h_L) + S_{L\eta} b_4 l_L] = 0; \\[6pt]
I_U(\theta_U'' + \kappa'') + m_U r_U h_U b_1 \kappa'' + m_U r_U (b_1 \xi_H'' - b_2 \eta_H'') \\
+ c_U(\theta_U + \varphi_U) + e_U l_U (S_{U\xi} b_1 - S_{U\eta} b_2) = 0; \\[6pt]
I_L(\theta_L'' - \kappa'') - m_L r_L h_L b_3 \kappa'' + m_L r_L (b_3 \xi_H'' + b_4 \eta_H'') \\
+ c_L(\theta_L + \varphi_L) + e_L l_L (S_{L\xi} b_3 + S_{L\eta} b_4) = 0; \\[6pt]
m_A \xi_A'' - e_U S_{U\xi} - e_L S_{L\xi} = 0; \\[6pt]
m_A \eta_A'' + m_a r_A \psi'' + m_a g - e_U S_{U\eta} - e_L S_{L\eta} = 0; \\[6pt]
I_A \psi'' + m_a r_A(\eta_A'' + \xi_A'' \psi + g) = 0,
\end{gathered}
\tag{12}
$$

where F_i are the generated forces:

$$
S_{U\eta} = \eta_H + h_U + l_U b_1 - \eta_A; \quad S_{L\eta} = \eta_H - h_L + l_L b_3 - \eta_A;
$$

$$
S_{U\xi} = \xi_H + h_U \kappa + l_U b_2 - \xi_A; \quad S_{L\xi} = \xi_H - h_L \kappa + l_L b_4 - \xi_A;
$$

$$
e_U = \frac{f(s_U - s_U^*)}{s_U s_U^*}; \quad e_L = \frac{f(s_L - s_L^*)}{s_L s_L^*}; \quad s_U = \sqrt{S_{U\xi}^2 + S_{U\eta}^2}; \quad s_L = \sqrt{S_{L\xi}^2 + S_{L\eta}^2}.
$$

The initial conditions of the problem are as follows:

$$
\begin{gathered}
t = 0, \quad \xi_s = 0; \quad \xi_H = \xi_{H0}; \quad \eta_H = \eta_{H0}; \quad \xi_A = \xi_{A0}; \quad \eta_A = \eta_{A0}; \\
\theta_U = \theta_{U0}; \quad \theta_L = \theta_{L0}; \quad \kappa = 0; \quad \psi = \psi_0; \quad \xi_s' = 0; \quad \xi_H' = 0; \quad \eta_H' = 0; \\
\xi_A' = 0; \quad \eta_A' = 0; \quad \theta_U' = 0; \quad \theta_L' = 0; \quad \kappa' = 0; \quad \psi' = 0,
\end{gathered}
\tag{13}
$$

where the constants η_{A0}, θ_{U0}, θ_{L0}, ξ_{H0}, η_{H0} are the solutions of the static problem (see the next section). Zero values of derivations correspond to the manner of archery technique (a breathing is stopped and a position is motionless).

According to the results of surface electromyography, the motor program of shooting is initiated before the signal to release can be heard. Thus it can be classified as an open loop. The anticipated balance release is characterized by an increasing activity of the *m. pectoralis major* only [1]. This muscle is activated to the balance changes of static equilibrium in the lateral plane. No muscles show significant changes of activity. Therefore, no generated forces should be present in the Lagrange equations (12), which describe the behaviour of the system in the vertical plane $F_i = 0$. The direction of arrow motion (the vector of the centre of mass) is described by the expressions:

$$\tan\zeta = \frac{\eta_A' + r_A\psi'}{-\xi_A'}; \quad \alpha = \psi - \zeta\,, \tag{14}$$

where ζ is the angle of projection, and α is the angle of attack. An arrow leaves a string just after its acceleration becomes equal to zero $(\xi_A'' = 0)$.

The system of equations (12) with initial conditions (13) presents the Cauchy problem for non-linear ordinary differential equations of the second order. It is impossible to obtain analytical solutions for the problem, therefore we used the Runge–
Kutta method applied in the program 'NDSolve' (Method Explicit Runge–Kutta) from the package of Mathematica 4.1 (www.wolfram.com).

This mathematical model describes dynamics of the bow and arrow system in the vertical plane (or, to be precise, bow and arrow common motion in the main plane of the bow). Indeed, the system performs 3D motion, but its displacement in the lateral plane is hundred times smaller than displacement in the main plane. The influence of the lateral motion on the main motion is negligibly small because of a significantly different value of displacement and the corresponding energy [7].

5. Static behaviour

Let us consider the character of the interaction between an archer and a bow at the instant of string release. In a modern technique of the sport of archery, you are not required to hold, but to grasp a bow handle. The centre of the mass of bow is located at the point of the interaction between the handle and the archer's hand. The mass of riser, stabilizers, sight and other rigid joints consists the main quarter of the whole bow mass. The mass of an arrow, a string, and movable elements of limbs consists about 3% of the whole mass of the system (www.huytusa.com). The corresponding gravity forces are negligibly small compared to the drawing force. So, the gravity forces have no influence on the bow shape and are compensated for the force of archer's hand. Therefore, in static problem, we can assume only two external forces, which act on the handle and on the string, i.e., the forces at the points *O* (the pivot point) and *A* (nock point), respectively (figure 3).

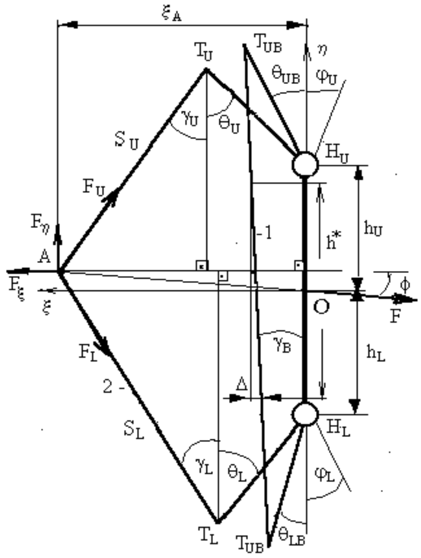

Fig. 3. Scheme of static model: 1 – braced bow, 2 – strung bow

The corresponding mathematical model of the bow in a given situation is represented by

$$\xi_A = l_U \sin\theta_U + s_U \sin\gamma_U; \quad \xi_A = l_L \sin\theta_L + s_L \sin\gamma_L;$$

$$\eta_A = h_U + l_U \cos\theta_U - s_U \cos\gamma_U; \quad \eta_A = s_L \cos\gamma_L - l_L \cos\theta_L - h_L;$$

$$\eta_A = h_U + l_U \cos\theta_U - s_U \cos\gamma_U; \quad \eta_A = s_L \cos\gamma_L - l_L \cos\theta_L - h_L; \tag{15}$$

$$c_U(\theta_U + \varphi_U) = F_U l_U \sin(\theta_U + \gamma_U); \quad c_L(\theta_L + \varphi_L) = F_L l_L \sin(\theta_L + \gamma_L);$$

$$F_\xi = -F_U \sin\gamma_U - F_L \sin\gamma_L; \quad F_\eta = F_U \cos\gamma_U - F_L \cos\gamma_L;$$

$$F_U = f\,\frac{s_U - s_U^*}{s_U^*}; \quad F_L = f\,\frac{s_L - s_L^*}{s_L^*}; \quad \tan\phi = \frac{F_\eta}{F_\xi}; \quad \tan\phi = \frac{\eta_A}{\xi_A},$$

where F_U and F_L are the forces in string branches; F_ξ and F_η are the projections of the drawing force onto the axes of coordinates; ξ_A is a drawing distance determined by the length of an arrow.

The initial conditions for the pivot point displacements during the bow hand and the handle interaction are given by:

$$\xi_{H0} = \frac{F_\xi}{c_{2\xi}}; \quad \eta_{H0} = \frac{F_\eta}{c_\eta}. \tag{16}$$

Mathematical model of the braced bow is represented by the following equations:

$$\begin{gathered} l_U \cos\theta_{UB} + l_L \cos\theta_{LB} + h_U + h_L = 2s_B \cos\gamma_B; \\ l_U \sin\theta_{UB} - l_L \sin\theta_{LB} = 2s_B \sin\gamma_B; \\ F_B l_U \sin(\theta_{UB} - \gamma_B) = c_U(\theta_{UB} + \varphi_U); \\ F_B l_L \sin(\theta_{LB} + \gamma_B) = c_L(\theta_{LB} + \varphi_L); \quad F_B = f\,\frac{s_B - s^*}{s^*}, \end{gathered} \tag{17}$$

where s_B is the string length and F_B is the force of the string stretching (see figure 3). The parameters of a bow with a braced string are marked with the subscript B.

The systems of equations (15) and (16) include transcendent and non-linear functions that cannot be solved analytically. Therefore to study of the static problem, we made use of a numerical method of iteration using the computer program 'Find' from the package of Mathcad 2000i Professional (www.mathcad.com).

6. Experimental modelling

Two kinds of limb models are popular in mechanical analysis of the modern sport bow, i.e., an elastic strip [4] and a rigid beam joined to a riser by an elastic element [3]. Both of the models have some advantages and limitations. As for the analysis of tensions and deformations of the limb and the influence of its shape on the bow efficiency, the first model is better. If we study the statics and dynamics of the bow as a whole system, the second model is more effective because of its simplicity and reasonable precision. Common parameters of the model (virtual length of a limb, centre of mass, moment of inertia and virtual stiffness) have been proposed by

HICKMAN [3]. The model was applied to a classic English bow. In order to use the model for a modern sports bow with recurved limb, we have to modify it.

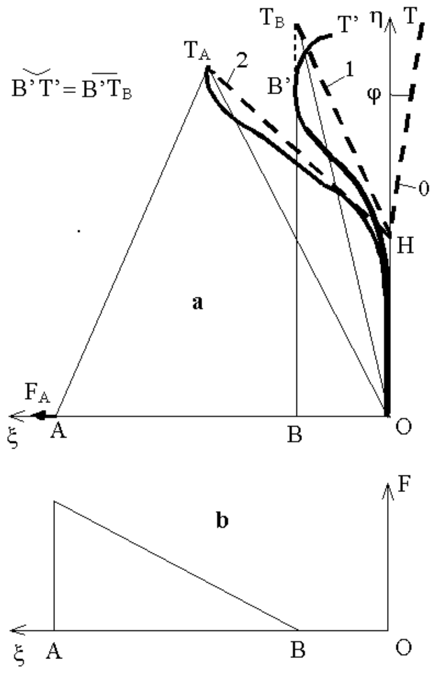

Fig. 4. Scheme for the transformation of the recurved limb to the Hickman model of the limb (a): 0 is a free situation (without a string), 1 – braced bow, 2 – strung bow; diagram of bow force vs. draw length (b)

We propose experimental and analytical method to determine the parameters of original model. Symmetrical shape of the bow in its main plane is studied in two positions (figure 4): the bow is braced with string (B), and the bow is conventionally strung (A). The break lines show a virtual part of a string and a limb, which transform the recurved bow scheme to the Hickman scheme. The string is in contact with the recurved part of the limb in the braced position between the points B' and T'. We assume the length of the curve ($B'T'$) to be equal to the length of the straight line ($B'T_B$).

The origin O of the Cartesian coordinate system is pinned to the handle in the middle of the bow. $O\xi$ is the axis of symmetry, axis $O\eta$ is based on the riser. Using simple algebraic equations (figure 4a)

$$(\eta_{T_A} - \eta_H)^2 + \xi_{T_A}^2 = l^2; \quad (\eta_{T_B} - \eta_H)^2 + \xi_{T_B}^2 = l^2$$

we arrive at two basic geometrical parameters of the model:

$$\eta_H = \frac{l_{OT_B}^2 - l_{OT_A}^2}{2(\eta_{T_B} - \eta_{T_A})}; \quad l = \sqrt{\frac{l_{OT_A}^2 + l_{OT_B}^2}{2} - \eta_H(\eta_{T_A} + \eta_{T_B} - \eta_H)}, \tag{18}$$

where l is a virtual length of the limb; $l_{OT_A} = OT_A$; $l_{OT_B} = OT_B$.

Using force, energetic and geometrical alignments (see figure 4a, b) :

$$E = \int_0^{\xi_A - \xi_B} F d\xi \approx \tfrac{1}{2} F_A(\xi_A - \xi_B); \quad E = c[(\theta_A + \varphi)^2 - (\theta_B + \varphi)^2]; \quad F_A = \frac{2c(\theta_A + \varphi)}{\eta_{T_A}},$$

we obtain the force parameters of the model:

$$c = \frac{F_A \eta_{T_A}}{2(\theta_A + \varphi)}; \quad \varphi = \frac{2k\theta_A - \theta_A^2 + \theta_B^2}{2(\theta_A - \theta_B - k)}, \tag{19}$$

where E, F_A are respectively the potential energy of the limbs and the force of a strung bow; c is a stiffness of the virtual elastic element instead of a distributed stiffness of a limb; $\theta_A = \angle \eta H T_A$; $\theta_B = \angle \eta H T_B$; φ is the angle of a virtual limb at free situation (i.e., without a string); $k = \dfrac{E}{F_A \eta_{T_A}}$. The role of another marks is explained in the scheme.

The recent types of sports bows are designed with nominal equal upper and lower limbs. Some previous types were designed with limbs of different shape and stiffness (www.yamaha.co.jp). The method mentioned above is suitable for determination of virtual parameters separately for the upper and lower limbs, too (see previous sections).

7. Practical application

Let us consider the modern sports bow and arrow made according to FITA (International Archery Federation) Standard (www.archery.org) labelled as: WIN&WIN Recurve Bow. The bow consists of Winact Riser (25”) and Long Limbs (70”), i.e., the bow handle length h^* is 635 mm (see figure 3) and the whole length of the bow (measured between tips of the limbs) reaches 1778 mm. The standard measure of bow asymmetry in the vertical plane named ‘tiller’ is $\Delta = 6$ mm and the bow force is F = 178 N. The arrow No. 2414 - 30” with 15% mass point was used.

Rated parameters of the bow are as follows (see the previous sections): $l_U = l_L =$ 531 mm; m_U = 106 g; m_L = 107 g; I_U = 68.1 kgcm^2; I_L = 68.3 kgcm^2; r_U = 227 mm; r_L = 228 mm; $c_U = c_L$ = 69.1 Nm; φ_U = 0.6047; φ_L = 0.6076; $h_U = h_L$ = 342 cm; m_H =

2.13 kg; I_H = 2128 kgcm^2; x_{CH} = –21 mm; y_{CH} = –34 mm; $s_U^* = 780$ mm; $s_L^* = 840$ mm; f = 255 N/cm; m_s = 7 g; $\xi_{A0} = 0.7576$.

We obtain the bow parameters in its braced position solving the system of equations (17): θ_{UB} = 0.4666; $\theta_{LB} = 0.4342$; s_B = 820 mm; γ_B = 0.00945; F_B = 316 N.

The arrow parameters are: l_a = 783 mm; m_a = 22.4 g; I_A = 73.6 kgcm^2; r_A = 510 mm.

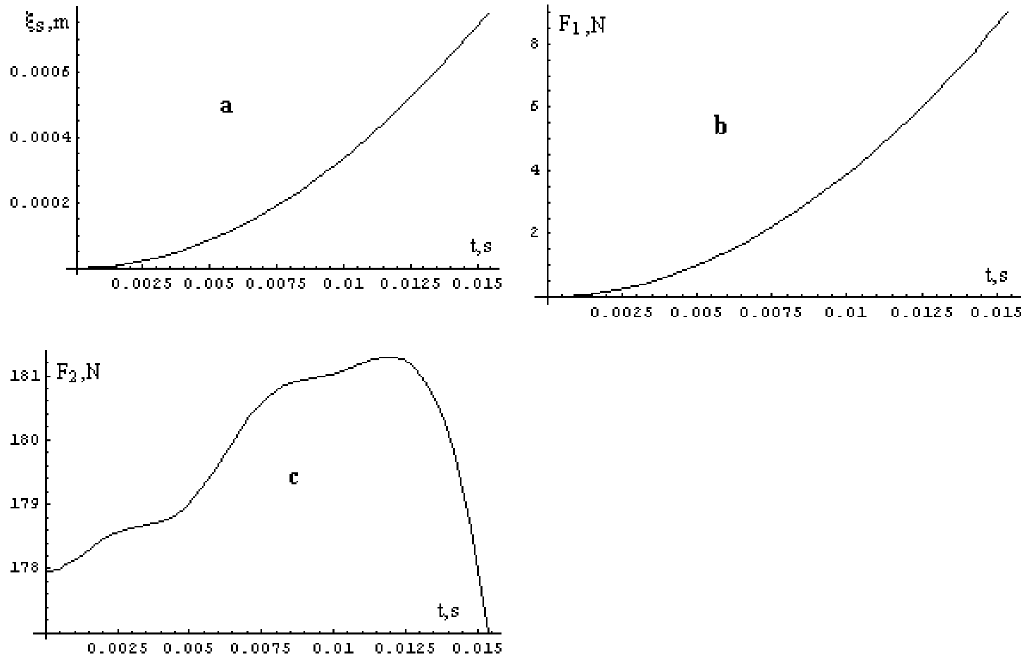

Fig. 5. Parameters of bow and archer interaction:
a – virtual displacement of the archer's body; b – recoil force acting on the body;
c – recoil force acting on the bow hand

The male archer's parameters are: $m_{1\xi}$ = 26 kg; $m_{2\xi}$ = 3.8 kg; m_η = 2.1 kg; $c_{1\xi}$ = 11.6 N/mm; $c_{2\xi}$ = 19.0 N/mm; c_η = 9.43 N/mm; $k_{1\xi}$ = 237 kg/s; $k_{2\xi}$ = 78 kg/s; k_η = 45 kg/s.

Solving the static problem (15), we arrive at the parameters under initial conditions (13) and (16):

η_{A0} = 42.6 mm; θ_{U0} = 0.7655; θ_{L0} = 0.7942; ξ_{H0} = 9.4 mm; η_{H0} = 1.1 mm. The other solutions are as follows: s_U = 786 mm; s_L = 846 mm; γ_U = 0.5189; γ_L = 0.4641; F_U = 186 N; F_L = 192 N; F_ξ = 178 N; F_η = 10 N.

The dynamics of archer, bow and arrow during their common motion, as an initial rest height $\eta_{P0} =$ 37 mm, is presented in figures 5, 6, 7.

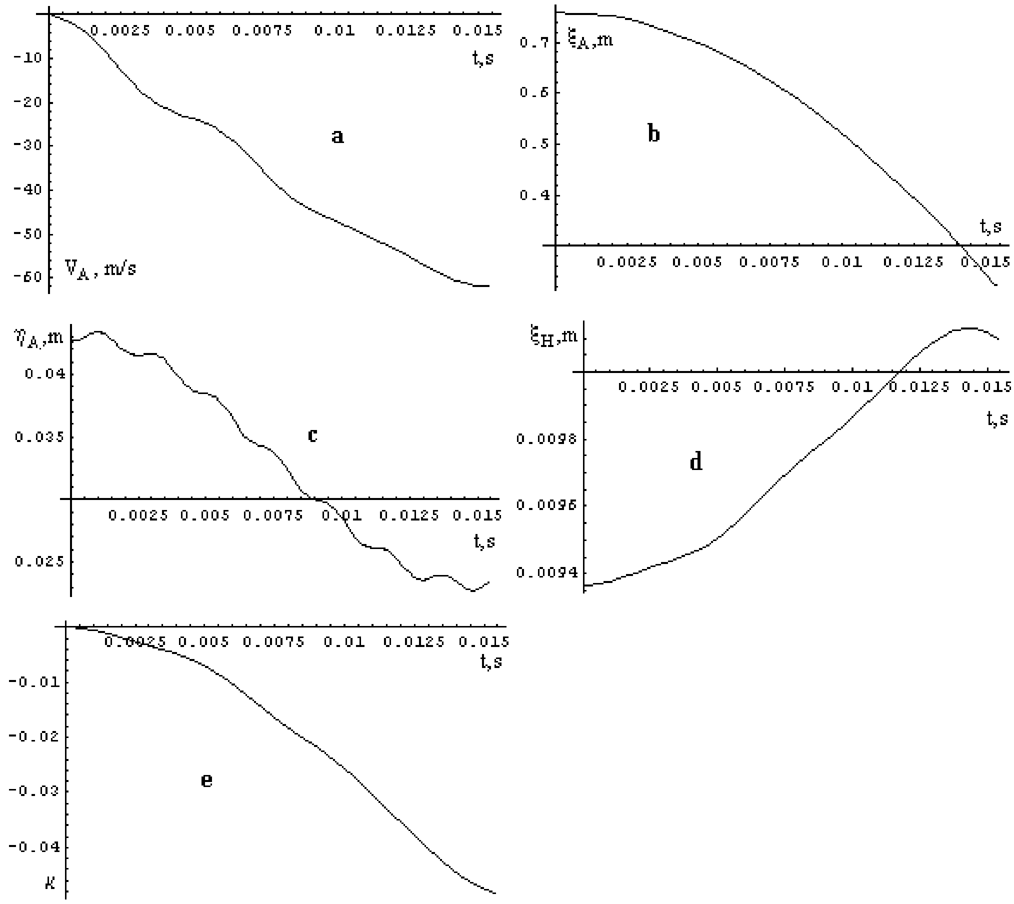

Fig. 6. Parameters of bow and arrow interaction:
a – longitudinal speed of an arrow; b – longitudinal displacement of an arrow;
c – perpendicular displacement of string and arrow nock point; d – longitudinal displacement of the pivot point (hand and handle); e – angular displacement of a handle

The time of bow and arrow common motion (from string release up to arrow shooting), i.e., about 0.0154 s, was calculated considering the instant of the maximum longitudinal speed (V_A = 62 m/s) to correspond directly to zero value of acceleration ($\xi_A'' = 0$, see figure 6 a).

A recoil force acting between a bow hand and a handle after string release varies in the range close to 3% of its initial (static) value (see figure 5c). A force, almost 20 times smaller (see figure 5b), is transferred to the ground via an archer's body, while its displacement remains near 1 mm (see figure 5a).

Because of the different character of static and dynamic balances of forces, an arrow presses on a string a few millimetres deeper ($\xi_A = 223$ mm, see figure 6b) compared to its braced position ($\xi_A = 231$ mm). Displacement of the pivot point, i.e., the point at which a bow hand and a handle are in contact, is about one millimetre and approximately is equal in the longitudinal ($9.4 < \xi_H < 10.2$ mm) and perpendicular directions (see figure 6d).

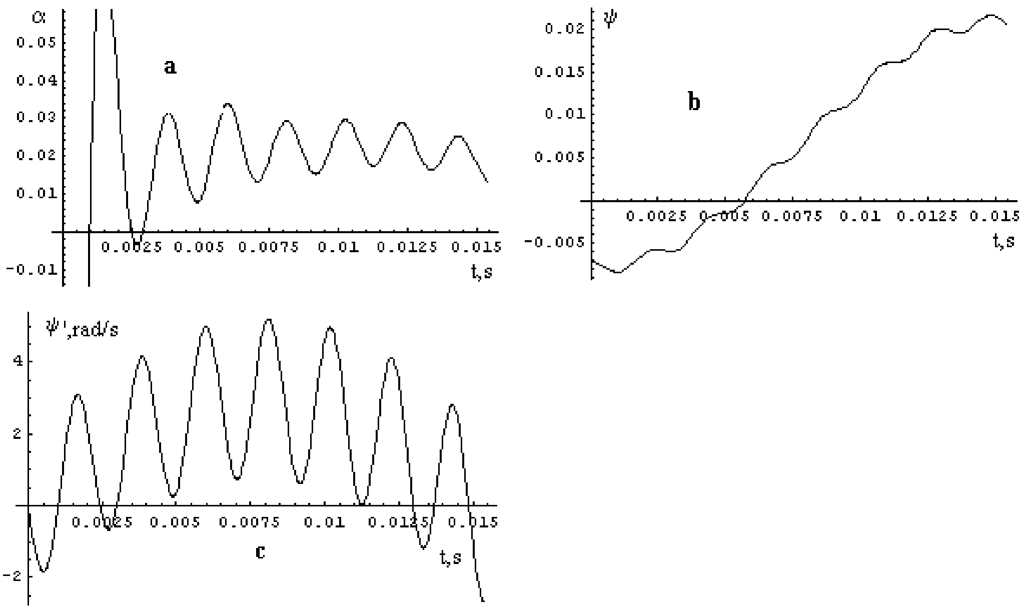

Fig. 7. Parameters of arrow internal ballistics: a – angle of attack of an arrow; b – attitude angle; c – angular speed of the arrow in the main plane

String and arrow common motion (internal ballistics) is accompanied by intensive oscillations, which are caused by the destruction of the static balance of forces at the instant of string release. There are seven full cycles of oscillations during the motion. The results of computer simulation make it possible to determine the bow and arrow adjusted parameters, which minimize the angle of attack and angular speed of an arrow that is better for a good shot (see figure 6).

8. Conclusions

1. The results of the modelling of the archer–bow–arrow system correlate with well-known results of high-speed video analysis: the process of common motion has a significant non-linear character.

2. A recoil force acting between a bow hand and a handle after string release varies in the range close to 3% of its initial (static) value. A force, almost 20 times smaller, is transferred to the ground via an archer's body, while its displacement remains near 1 mm.

3. Because of the different character of static and dynamic balances of forces, an arrow presses on a string a few millimetres deeper compared to its braced position.

4. String and arrow common motion (internal ballistics) is accompanied by intensive oscillations, which are caused by the destruction of the static balance of forces at the instant of string release. There are full seven cycles of oscillations during the motion.

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